On minimal shapes and topological invariants in hyperbolic lattices

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Abstract

We characterize the set of finite shapes with minimal perimeter on hyperbolic lattices given by regular tilings of the hyperbolic plane whose tiles are regular p-gons meeting at vertices of degree q, with $1/p + 1/q < \frac{1}{2}$. The main tool is a layer decomposition due to Rietman–Nienhuis–Oitmaa and Moran, which allows us to prove convergence towards the Cheeger constant when these shapes exhaust the lattice. Furthermore, we apply a celebrated result of Floyd–Plotnick, which will allow us to compute the Euler characteristic for these graphs in terms of certain growth functions and the number of n-sized animals on those lattices.

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1 Introduction

Hyperbolic lattices $\mathcal{L}_{p,q}$ play a crucial role in geometry, topology, and mathematical physics when considering spaces beyond the Euclidean setting, for example by including negative constant curvature in the space. Some concrete applications can be found in e.g. crystallography [2], non-Euclidean analog of the quantum spin Hall effect [17] or quantum electrodynamics [8], with remarkable experimental consequences [3].

These lattices are discrete symmetry groups acting on the hyperbolic plane, forming regular tilings or tessellations of the two-dimensional space with constant Ricci curvature of -1 like the Poincaré disc. Moreover, they form the simplest examples of regular lattices (p vertices of degree q) associated to a non-Euclidean geometric setting and have been constructed through a layer decomposition in [13]. For examples, see Figures 1.1 and 1.2. In [13], the authors observed that these lattices can alternatively be constructed through successive layers of tilings, starting from a

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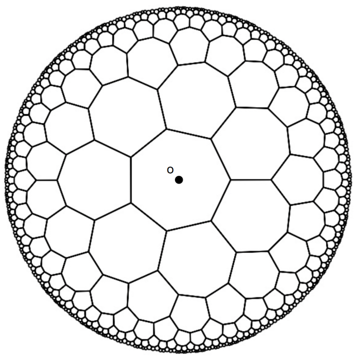

Figure 1.1: Embedding of $\mathcal{L}_{7,3}$ in the hyperbolic disc

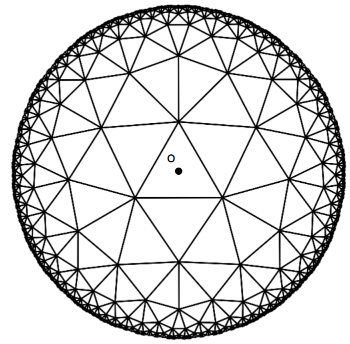

Figure 1.2: Embedding of $\mathcal{L}_{3,7}$ in the hyperbolic disc

fundamental tiling, where the vertices in each layer follow a simple recursive pattern. This recursive structure was then explicitly used in [11] to compute the growth rate of these lattices.

Invariants are quantities or properties that remain unchanged under certain transformations. In the context of hyperbolic lattices, they can be algebraic, geometric, or topological, and they help characterize the intrinsic properties of these structures. In this work, we will be specifically interested in two topological invariants, namely the *Cheeger or isoperimetric constant and the Euler characteristic* of hyperbolic lattices.

Isoperimetric constants $i_e(\cdot)$ are important because they quantify how efficiently a shape encloses an area (in two dimensions) relative to its perimeter, and because they quantify global connectivity in a graph. Intuitively, the isoperimetric constant of a graph represents the measure of how difficult it is to separate the graph into two parts. A high isoperimetric constant implies that every finite subset of vertices has many edges connecting it to the rest of the graph, indicating strong connectivity. Conversely, if the isoperimetric constant is zero, the graph contains finite subsets with very few boundary edges, suggesting a structure that can be easily separated (such as $\mathbb{Z}^d$ which are examples of amenable graphs).

The 1-skeleton of hyperbolic lattice provides an example of an infinite non-amenable graph G, i.e. $i_e(G) > 0$, meaning that no finite subset has a *small boundary* relative to its size. The positivity of $i_e(G)$ explains why these graphs (and models defined on these graphs) behave differently from their counterparts on classical structures like $\mathbb{Z}^d$.

For example, percolation follows a distinct pattern on these lattices [10, 6], as e.g. there are two percolation thresholds where one has regimes in which there are 0, infinitely many infinite clusters or 1 infinite cluster, analogously to what happens for percolation on $\mathbb{T}_d \times \mathbb{Z}$, see also [5].

The ferromagnetic nearest-neighbors Ising model on these lattices also behaves much differently from its counterpart on $\mathbb{Z}^2$ at low temperature. In the latter setting, Aizenmann [1] and Higuchi [7] proved that the set of *extremal* Gibbs states at low temperature consists of two measures. For hyperbolic lattices, it is now known that there are uncountably many extremal Gibbs states at low temperature [4], indexed by certain bi-infinite geodesics on the dual lattice (similar to Dobrushin interfaces), providing a positive answer to a broader conjecture by Series–Sinai [15] in this special case.

Another important question related to the isoperimetric problem is what are the minimal shapes associated with the minimal perimeter given a fixed volume and whether they realize the isoperimetric constant. For example, in $\mathbb{Z}^d$ minimal shapes are squares and in $\mathbb{R}^d$ balls.

In [6, Theorem 4.1] the authors compute the value of the isoperimetric constant $i_e(G)$ for the hyperbolic lattice and mention in [6, Remark 4.3] that *the combinatorial balls do not realize the isoperimetric constants*. The authors consider combinatorial balls in the dual lattice $\mathcal{L}'_{p,q} = \mathcal{L}_{q,p}$ starting from a vertex $\mathbf{o}$.
In this paper, we prove the following results:

Theorem. [Shapes of minimal perimeter] *For a fixed volume $N \in \mathbb{N}$, there exists an explicit set of shapes with strict minimal perimeter $\mathcal{M}_N$.*

In particular, the ratio between the external boundary and the number of vertices in these shapes converges towards $i_e(\mathcal{L}_{p,q})$, when the shape exhausts $\mathcal{L}_{p,q}$. This theorem provides an answer to a long-standing open question.

Theorem. [Häggström–Jonasson–Lyons constant via limits] *For $i_e(\mathcal{L}_{p,q})$ computed in [6, Theorem 4.1] we have that*

$$i_e(\mathcal{L}_{p,q}) = \lim_{|\mathscr{M}|\to\infty} \frac{|\partial_e \mathscr{M}|}{|\mathscr{M}|},$$

where $\mathscr{M} \in \mathcal{M}_N$.

The Euler characteristic χ is a fundamental invariant in topology and combinatorics that provides insight into the structure of shapes, surfaces, and graphs. Originally introduced by Leonhard Euler in the context of polyhedra, it has since found applications in graph theory, network analysis, and computational topology. For finite graphs it is equal to the number of vertices and number of faces minus the number of edges. Applications can be found for example in topological data analysis [16] or network analysis [9]. Intuitively, the Euler characteristic describes how well connected a graph or surface is. A positive χ means that the graph is simply connected whereas a negative χ indicates that it has holes. For infinite graphs it is not obvious how to define the Euler characteristic. A pioneering work by Floyd–Plotnick [12] relates the growth functions associated on these Fuchsian groups to their Euler characteristic. We will use their result to prove that:

Proposition. [Euler characteristic] *Let $\mathcal{G}_{p,q}$ be the Fuchsian group associated to $\mathcal{L}_{p,q}$. Then its Euler characteristic is given by*

$$\mathcal{X}(\mathcal{G}_{pq}) = \frac{2q + 2p - pq}{2p}.$$

In the Corollary 3.4 we compute the explicit number of animals of size n in the graph $\mathcal{L}_{p,q}$.

The structure of the paper is as follows. In Section 2 we will define all objects and fix notation, Section 3 contains all results which are proven in Section 4.

2 Definitions and notations

Consider a general graph $G = (V, E)$ and let $A \subset V$ be a subset of vertices. We denote by $|A|$ the cardinality of A and by $A^c = V \setminus A$ the complement of A in V. Two vertices are connected if there exists an edge between them. The set A is called *connected*, if for each $v, u \in A$ there exists a sequences of connected vertices in $w_1, ..., w_n \in A$ such that $w_1 = v$ and $w_n = u$. The *perimeter* $|\partial_e A|$ of the set A is the cardinality of the external boundary of A, defined as

$$|\partial_e A| := |\{(v, w) \in E \,|\, v \in A,\, w \notin A\}|. \tag{2.1}$$

Definition 2.1. *(i) (Finite graphs) The Cheeger (or isoperimetric) constant i_m, resp. the Cheeger geometric constant i_m^g, are defined as*

$$i_m(G) = \min_{\substack{A \subset V, \\ |A|=m}} \frac{|\partial_e A|}{|A|} \qquad \text{and} \qquad i_m^g(G) = \min_{\substack{A \subset V, \\ |A|=m}} \frac{|\partial_e A|}{vol(A)}, \tag{2.2}$$

where $vol(A)$ is the sum of the degrees of the vertices, i.e. $vol(A) = \sum_{v \in A} deg(v)$.

(ii) (Infinite graphs) The Cheeger constant of G is defined as follows

$$i_e(G) = \inf_{\substack{A \subset V, \\ 0<|A|<\infty}} \frac{|\partial_e A|}{|A|}. \tag{2.3}$$

We will define the hyperbolic lattice using the construction from [13]. Let $\{p, q\}$ be two integers such that $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$. $\mathcal{G}_{p,q}$ is the *Fuchsian group* defined as follows

$$\mathcal{G}_{p,q} := \langle a, b | a^p, b^q, (ab)^2 \rangle \tag{2.4}$$

where a denotes the rotation around a given lattice point over an angle $\alpha = 2\pi/q$ and b a rotation around the center of an adjacent face over an angle $\beta = 2\pi/p$. The rotation is defined w.r.t. the hyperbolic metric (remark that we fix the scalar curvature to -1 with this choice)

$$ds^2 = \frac{4(dx^2 + dy^2)}{(1 - x^2 - y^2)^2}.$$

The group $\mathcal{G}_{p,q} \subset PSU(1,1)$ is a subgroup of the group of isometries of the unit disc in the complex plane. A representation ρ of a, b can be defined in the following way:

$$\rho(a) = \pm \begin{pmatrix} e^{i\beta/2} & 0 \\ 0 & e^{-i\beta/2} \end{pmatrix}$$
$$\rho(b) = \pm \frac{1}{1 - r^2} \begin{pmatrix} e^{i\alpha/2} - r^2 e^{-i\alpha/2} & -r(e^{i\alpha/2} - e^{-i\alpha/2}) \\ r(e^{i\alpha/2} - e^{-i\alpha/2}) & e^{-i\alpha/2} - r^2 e^{i\alpha/2} \end{pmatrix}$$

and $r^2 = \frac{\cos((\alpha+\beta)/2)}{\cos((\alpha-\beta)/2)}$.

Definition 2.2. *The hyperbolic lattice $\mathcal{L}_{p,q} = (\mathcal{V}, \mathcal{E})$ is defined as follows. Choose $z = \mathbf{o}$ as the center of the fundamental face and choose $z = r$ an adjacent lattice point. The vertices are generated by words a, b acting on $z = r$. Edges are drawn between points as a result of the action of $g_1, g_2 \in \mathcal{G}_{p,q}$ on r if $g_2 = g_1 a^n b a^m$ for some $m, n = 0, \ldots, p-1$.*

Note that the faces are equilateral and the lattice $\mathcal{L}_{p,q}$ is naturally embedded into the Poincaré disc, see Figures 1.1, 1.2.

In the following, we will describe an alternative way to construct $\mathcal{L}_{p,q}$ in terms of layers. Let $k \in \mathbb{N} \cup \{0\}$, we define the k-th *layer*, denoted by L_k, as the set of vertices in $\mathcal{V}$ constructed in the following way.

The zero layer L_0 is the set of p vertices in the unique face of $\mathcal{L}_{p,q}$ containing $\mathbf{o}$. The first layer L_1 is the set of vertices (not in L_0) of all the faces which are adjacent with the face containing $\mathbf{o}$ (including those sharing just a vertex). For $k \geq 2$, we define L_k iteratively as the set of vertices of all the faces which are adjacent with the face containing the vertices in L_{k-1}. Denote by $B_k(\mathbf{o}) \coloneqq \bigcup_{l=0}^{k} L_l$ the ball of radius k centered at $\mathbf{o}$, see Figure 2.1 for an example. If the layer depends on a different reference point x (such as the middle point of a tile), we will write $L_k(x)$.

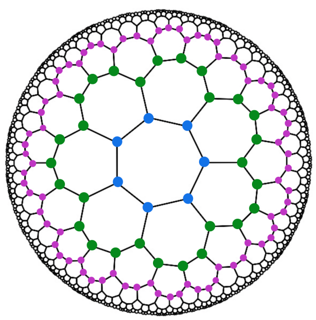

Figure 2.1: Example of layers L_0, L_1, L_2. The union is the ball $B_2(o)$.

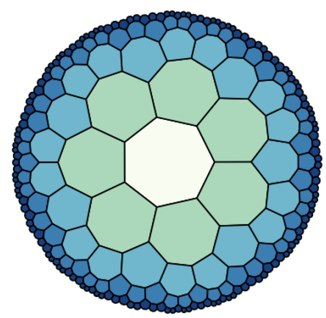

Figure 2.2: Example of "combinatorial balls" defined via tilings in [12].

Clearly, $\mathcal{V} = \cup_{k\geq 0} L_k$ and $L_j \cap L_k = \emptyset$ if $j \neq k$. Let n be fixed and consider first $p \geq 4$. We will distinguish between two types of vertices in each L_n in a recursive way, those which are connected to vertices from the previous layer L_{n-1}, denoted by $I_{n;p,q}$ (from "internal vertices") and those which are not, denote by $E_{n;p,q}$ (from "external vertices"), see also the construction in [13]. More precisely,

- $I_{n;p,q} \coloneqq \{v \in L_n \mid \exists\, w \in L_{n-1} : (v, w) \in \mathcal{E}\}$;
- $E_{n;p,q} \coloneqq L_n \setminus I_{n;p,q}$,

so that $|I_{n;p,q}| + |E_{n;p,q}| = |L_n|$. See Figure 2.3 for an example. For $p = 3$ (triangulations) the situation is special since all vertices in a layer are connected to the previous layer. In this case, we will define the following subsets for $n \geq 2$:

- $I'_{n;3,q} = \{v \in L_n \mid \exists\, w \in L_{n-1} : (v, w) \in \mathcal{E}\}$;
- $I''_{n;3,q} = \{v \in L_n \mid \exists\, w_1, w_2 \in L_{n-1} : (v, w_1), (v, w_2) \in \mathcal{E}\}$,

see Figure 2.4. In this case, $|I'_{n;3,q}| + 2|I''_{n;3,q}| = |L_n|$. The first set consists of the 3 points in the first triangle L_0.

We will add the dependence on a reference point x in the definition of the sets $I_{n;p,q}(x)$, $E_{n;p,q}(x)$, $I'_{n;3,q}(x)$ resp. $I''_{n;3,q}(x)$ if $x \neq \mathbf{o}$.

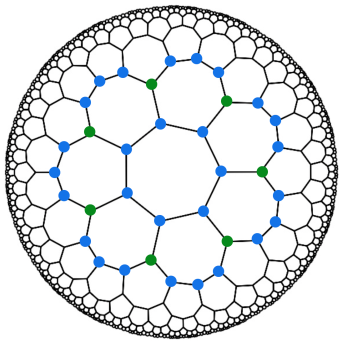

Figure 2.3: Example of the sets $E_{0;7,3}$ and $E_{1;7,3}$ in blue and $I_{1;3,7}$ in green.

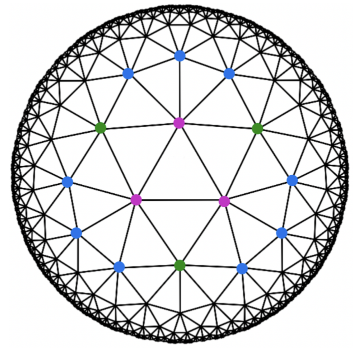

Figure 2.4: Example of the sets $I'_{1;3,7}$ in blue, $I''_{1;3,7}$ in green and the set L_0 in pink.

Remark 2.3. Note that the "combinatorial balls" $\mathcal{B}_n$ constructed in [12] using the word norm on the Fuchsian group $\mathcal{G}_{p,q}$ are unions of tiles. If we look at the corresponding combinatorial ball in the dual lattice $\mathcal{B}'_n$ of radius n from the origin $\mathbf{o}$ defined in [6], then $|\partial_e \mathcal{B}'_n| < |\partial_e B_n|$ and $|\partial_e \mathcal{B}'_n|/|\mathcal{B}'_n| > |\partial_e B_n|/|B_n|$. This is the reason why their combinatorial balls will not realize the isoperimetric constant (see Remark 4.3 in [6]) and ours will. For an example comparison we refer to Figures 2.5 vs 2.6.

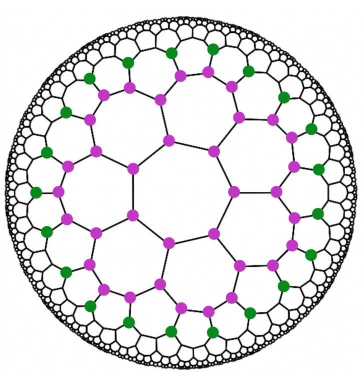

Figure 2.5: Example of $B_1(\mathbf{o})$ in pink and the green vertices are indicating $\partial_e B_1(\mathbf{o})$. We have $|\partial_e B_1(\mathbf{o})|/|B_1(\mathbf{o})| = 3/5$.

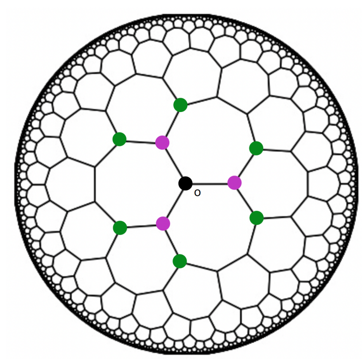

Figure 2.6: Example of $\mathcal{B}_1(\mathbf{o})$ in pink and the green vertices are indicating $\partial_e \mathcal{B}_1(\mathbf{o})$. We have $|\partial_e \mathcal{B}_1(\mathbf{o})|/|\mathcal{B}_1(\mathbf{o})| = 2$.

For any connected set of vertices A, we will define the smallest combinatorial ball containing A and a largest ball contained in A.

Definition 2.4. *Let $A \subset \mathcal{V}$ be any connected set of vertices (order them lexicographically $\preceq$), and set $|A| = N$. We define $B_{A,max}$ the largest ball contained in A as follows. If the vertices in A do not form a polygon (or a tile) then set $B_{A,max} = \emptyset$. Otherwise, there exist $M \leq N - p$ layers $L_1(x_i)$ in A with middle points $x_1, \ldots, x_M$. Let*

$$\{\overline{x}, \overline{m}\} := \underset{l \in \mathbb{N}}{arg\,max}\ \underset{\{x_1,\ldots,x_M\}}{arg\,max} \left\{ \bigcup_{k=0}^{l} L_k(x_i) : \bigcup_{k=0}^{l} L_k(x_i) \subset A \right\}.$$

Then $B_{A,max}$ is defined as

$$B_{A,max}(\overline{x}) := \bigcup_{k=0}^{\overline{m}} L_k(\overline{x}).$$

Note that this ball is not uniquely defined. In the case that several sets $B_{A,max}(\overline{x})$ can be constructed in this way we take the ball with the smallest reference point in lexicographic order.

Definition 2.5. *Given $\overline{x}$ as in Definition 2.4, let*

$$\overline{M} = \underset{l \in \mathbb{N}}{arg\,min} \left\{ \bigcup_{k=0}^{l} L_k(\overline{x}) : \bigcup_{k=0}^{l} L_k(\overline{x}) \supset A \right\}.$$

We define the minimal ball containing A, by

$$B_{A,Min}(\overline{x}) = \bigcup_{k=0}^{\overline{M}} L_k(\overline{x}).$$

An example can be found in Figure 2.7.

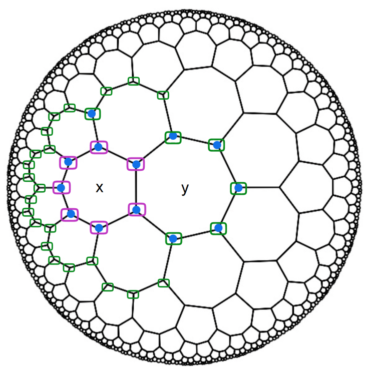

Figure 2.7: Example of the set of connected points A, displayed as blue points. There are two tiles present in A. One is centered at x and one at y with $x \preceq y$. The ball $B_{A,max}(x)$ is displayed by the pink circles and and the layer $B_{A,Min}(x) \setminus B_{A,max}(x)$ by the green circles.

Furthermore we will characterize the vertices v in the layers in $B_{A,Min}(\overline{x}) \setminus B_{A,max}(\overline{x})$ as empty if $v \notin A$ and occupied if $v \in A$. A sequence of consecutive empty, resp. occupied, vertices in the same layer is called a *strip*.

Finally, for a fixed $N \in \mathbb{N}$ and connected set $A \subset \mathcal{V}$ such that $|A| = N$ we will define sets $\mathcal{M}_N$ which will turn out to be the set of minimal shapes which have minimal perimeter.

Definition 2.6. *Fix $N \in \mathbb{N}$ and let A be a connected set $A \subset \mathcal{V}$ such that $|A| = N$. We call*

$$\mathcal{N}_e = (B_{A,Min}(\overline{x}) \setminus B_{A,max}(\overline{x})) \cap A^c, \quad resp.\ \mathcal{N}_o = (B_{A,Min}(\overline{x}) \setminus B_{A,max}(\overline{x})) \cap A$$

the set of empty, resp. occupied vertices, in $B_{A,Min}(\overline{x}) \setminus B_{A,max}(\overline{x})$. Define a strip S of length $|\mathcal{N}_o|$ in some layer L_K for K large enough. Denote by o_{max} the maximal possible number of vertices $v \in S$ which are also in $I_{K;p,q}$ for $p \geq 4$, resp. also in $I''_{K;3,q}$ for $p = 3$. Moreover let s_e denote the number of empty strips in the layers of $B_{A,Min}(\overline{x}) \setminus B_{A,max}(\overline{x})$. Then $A \in \mathcal{M}_N$ if it satisfies one of the following conditions:

(C1) $s_e = 0$ *and* $B_{A,Min}(\overline{x}) = B_{A,max}(\overline{x})$*.*

(C2) $s_e \geq 1$ *and the set* $\mathcal{N}_o$ *contains precisely* $o_{max} + (s_e - 1)$ *vertices* v *such that* $v \in \bigcup_{r=\overline{m}+1}^{\overline{M}} I_{r;p,q}$ *resp.* $v \in \bigcup_{r=\overline{m}+1}^{\overline{M}} I''_{r;3,q}$*.*

We construct some examples of sets in $\mathcal{M}_{17}$ in Figures 2.8 and 2.9.

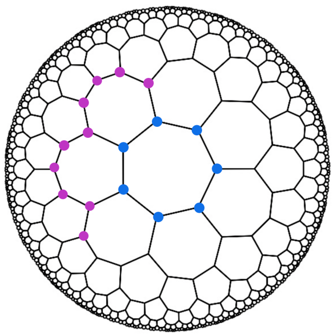

Figure 2.8: Example of $A \in \mathcal{M}_{17}$ with $|\mathcal{N}_o| = 10$, $|\mathcal{N}_e| = 18$. We have $s_e = 1$, $m = 3$, and three vertices are in $I_{1;7,3}$.

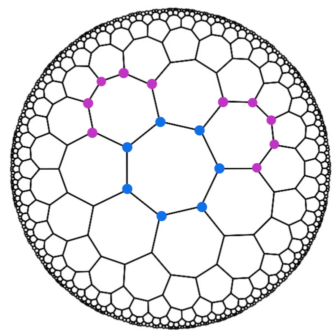

Figure 2.9: Example of $A \in \mathcal{M}_{17}$ with $|\mathcal{N}_o| = 10$, $|\mathcal{N}_e| = 18$. We have $s_e = 2$, $m = 3$, and $3 + (2 - 1)$ vertices are in $I_{1;7,3}$.

3 Results

In this section, we present the main results of this paper. The first result concerns characterizing the sets of minimal perimeter for a fixed volume $N \in \mathbb{N}$ for any hyperbolic lattice $\mathcal{L}_{p,q}$.

Theorem 3.1 (Minimal perimeter)**.** *Given* $N \in \mathbb{N}$*, let* $\mathcal{S}_N = \{A \subset \mathcal{V} : |A| = N\}$ *be the set of all subsets of vertices with size* N*. Then for any* $A \in \mathcal{S}_N \setminus \mathcal{M}_N$*:*

$$|\partial_e A| > |\partial_e \mathscr{M}|,$$

where $\mathscr{M} \in \mathcal{M}_N$*. Moreover, given a finite graph* G *with vertex set* $B_N(\mathbf{o})$*, the Cheeger and geometric Cheeger constants are equal to*

$$i_N(G) = \frac{|\partial_e \mathscr{M}|}{N} \qquad \text{and} \qquad i_N^g(G) = \frac{|\partial_e \mathscr{M}|}{qN},$$

where i_N, i_N^g *are defined in Definition 2.1.*

In the following theorem we will show that our minimal shapes in $\mathcal{M}_N$ realize the Cheeger constant (or isoperimetric constant) computed in [6, Theorem 4.1], when $N \to \infty$.

Theorem 3.2 (Realizing the Cheeger constant)**.** *For some* $N \in \mathbb{N}$*, let* $\mathscr{M} \in \mathcal{M}_N$ *and the Cheeger constant be equal to*

$$i_e(\mathcal{L}_{p,q}) = (q-2)\sqrt{1 - \frac{4}{(p-2)(q-2)}}.$$

Then

$$i_e(\mathcal{L}_{p,q}) = \lim_{|\mathscr{M}| \to \infty} \frac{|\partial_e \mathscr{M}|}{|\mathscr{M}|}. \tag{3.1}$$

Proposition 3.3. *(Euler characteristic) Let $\mathcal{G}_{p,q}$ be the Fuchsian group defined in Equation 2.4 associated to $\mathcal{L}_{p,q}$ and $\{p,q\}$ such that $\frac{1}{p}+\frac{1}{q}<\frac{1}{2}$. Then the Euler characteristic is equal to*

$$\mathcal{X}(\mathcal{G}_{pq}) = \frac{2q+2p-pq}{2p}.$$

From the proof of Proposition 3.3 we have the following interesting corollary which provides a formula for the number of connected sets of size n.

Corollary 3.4. *Let $g_p(z)=\sum_{n=0}^{\infty} a_n^{(p)} z^n$ be the generating function for the number of connected components of size n depending on p. Then, we have that*

$$a_n^{(p)} = \begin{cases} c_{p,q}(C_+\lambda_+^n + C_-\lambda_-^n) & \text{if } p\geq 4 \\ 3+\tilde{C}_+\lambda_+^{n-1}+\tilde{C}_-\lambda_-^{n-1} & \text{if } p=3 \end{cases}$$

where $c_{p,q}=\frac{p}{2\sqrt{(p-2)(p(q-2)-2q)}}$, the other constants $C_\pm, \tilde{C}_\pm$ are equal to

$$C_\pm = (\sqrt{(p-2)(p(q-2)-2q)} \mp 2\sqrt{q-2} \pm p\sqrt{q-2}) \tag{3.2}$$

$$\tilde{C}_\pm = \frac{3}{2}\left(q-3\pm\frac{(q-5)(q-2)}{\sqrt{(q-6)(q-2)}}\right), \tag{3.3}$$

and $\lambda_\pm$ are the eigenvalues equal to $\frac{1}{2}\left(2+p(q-2)-2q\pm\sqrt{(p-2)(q-2)(q(p-2)-2p)}\right)$.

We postpone the proofs of the previous results to the next Section 4.

4 Proofs

In this Section we will prove our main results.

4.1 Proof of Theorem 3.1

In order to prove Theorem 3.1 we need the following lemma. In this lemma we show that sets A which are not connected cannot be in $\mathcal{M}_N$.

Lemma 4.1. *Let $A\subset\mathcal{V}$ be a non-connected set of vertices, and $|A|=N$. Then there exists a set $B\subset\mathcal{V}$ such that $|B|=|A|$ and $|\partial_e B|<|\partial_e A|$.*

Proof of Lemma 4.1. Suppose that A is composed by m connected components $A_1,...,A_m$ with $m>1$. W.l.o.g. let A_2 be the closest (in graph distance) connected set of vertices to A_1 in the same layer. Let B be set of vertices composed of $m-1$ connected components $B_1,...,B_{m-1}$ such that $B_i=A_i$ for $i=3,...,m-1$ and $B_1=A_1\cup H(A_2)$ is a connected set of vertices where $H(A_2)$ is the homomorphism translating vertices from A_2 towards A_1. Thus, $|\partial_e B|\leq|\partial_e A|-1<|\partial_e A|$ and we conclude. □

In the following, we will prove Theorem 3.1. Given any set $D\in\mathcal{S}_N\setminus\mathcal{M}_N$ we will construct a set A out of D by elementary operations such that $|\partial_e D|>|\partial_e A|$. By Lemma 4.1 we can exclude all sets D which are not connected. Given a connected set D, we construct the balls $B_{D,Min}(\overline{x})$

resp. $B_{D,max}(\overline{x})$ containing resp. contained in D, see also Definitions 2.4, resp. 2.5. Recall that we can write any such D as

$$D = B_{D,max} \cup \mathcal{N}_o.$$

To ease notation we will assume w.l.o.g. that $\overline{x} = \mathbf{o}$. The strategy will be to show that any arrangement of occupied/empty strips of vertices in the annulus $B_{D,Min} \setminus B_{D,max}$ which is satisfying conditions (C1) or (C2) in Definition 2.6, has larger perimeter. We will develop the proof for the case $p \geq 4$ and leave $p = 3$ for the reader since it is a simple adaptation replacing the set I by I''.

In the following argument we will construct a set of smaller perimeter than D depending on the number of empty strips s_e in the annulus. Recall that if $s_e = 0$, then $D \in \mathcal{M}_N$ which is a contradiction. We will distinguish between three cases:

(I) $s_e = 1$ and the empty strip is in the last layer, $S_e \in L_{\overline{M}}$.

(II) $s_e = 1$ and the empty strip is not in the last layer, $S_e \in L_k$ where $k < \overline{M}$.

(III) Several empty strips, $s_e > 1$.

`CASE (I)` Note that in this case $B_{D,Min} = B_{D,max} \cup L_{\overline{M}}$. Call S_o the occupied strip in $L_{\overline{M}}$, in this case we have that $S_o = \mathcal{N}_o$. Then we know that its vertices belong either to $I_{\overline{M};p,q}$ resp. to $E_{\overline{M};p,q}$. In fact, we can decompose a general strip S_o, as

$$S_o = (S_o \cap I_{\overline{M};p,q}) \cup (S_o \cap E_{\overline{M};p,q}) \coloneqq S_o(I) \cup S_o(E), \tag{4.1}$$

and $|S_o(I)| < o_{max}$ by assumption, see also Figure 4.1 for an example. If $|S_o(I)| = o_{max}$, then $D \in \mathcal{M}_N$ which is a contradiction. Recall that o_{max} was the maximal number of vertices in $I_{\overline{M};p,q}$ an occupied strip of size $|\mathcal{N}_o|$ can have. By a direct computation, we obtain

$$|\partial_e B_{D,max}| \leq |\partial_e D| + (|S_o(E)| + 2)(4 - q) + (|S_o(I)| - 2)(6 - q). \tag{4.2}$$

We will construct A as follows, an example can be found in Figure 4.2. Let S'_o denote an occupied strip in layer $L_{\overline{M}}$ such that $|S'_o| = |S_o|$ and $|S'_o(I)| = o_{max}$. Since $|S_o(I)| < o_{max}$, we then necessarily have that $|S_o(E)| > |S'_o(E)|$ and

$$o_{max} = |S_o(I)| + |S_o(E)| - |S'_o(E)|. \tag{4.3}$$

Set now $A \coloneqq B_{D,max} \cup S'_o$. Note that the construction of the set A is not unique. Therefore,

$$\begin{aligned}
|\partial_e A| &= |\partial_e B_{D,max}| + (|S'_o(E)| + 2)(q - 4) + (m - 2)(q - 6) \\
&\overset{(4.2)}{\leq} |\partial_e D| + (|S'_o(E)| + 2)|(q - 4) - (|S_o(E)| + 2)(q - 4) + (|m| - 2)(q - 6) - (|S_o(I)| - 2)(q - 6) \\
&= |\partial_e D| - (|S_o(E)| - |S'_o(E)|)(q - 4) + (o_{max} - |S_o(I)|)(q - 6) \\
&\overset{(4.3)}{=} |\partial_e D| - 2\underbrace{(o_{max} - |S_o(I)|)}_{>0 \text{ by assumption}} < |\partial_e D|.
\end{aligned} \tag{4.4}$$

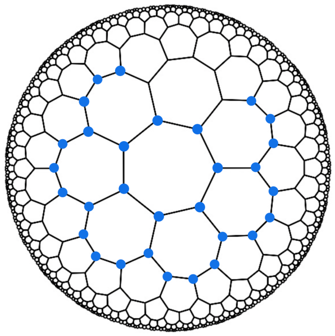

Figure 4.1: Example of $D \in \mathcal{S}_{30}$ with one empty and one occupied strip in the last layer, $|S_o(I)| = 5$.

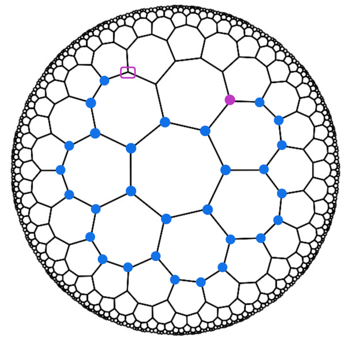

Figure 4.2: Example of A constructed from D by defining a new strip S' such that $|S'_o(I)| = o_{max} = 6$.

`CASE (II)` In this case we have one empty strip S_e lying in L_k for some $k < \overline{M}$. If $q \geq 4$ we can proceed as follows. First note that for a set $A' \coloneqq D \cup (S_e \setminus \{v\})$, where we added a vertex to the empty strip, the perimeter satisfies

$$|\partial_e A'| \leq |\partial_e D| + (2 - q). \tag{4.5}$$

Construct A by removing an occupied vertex w from the last layer in $E_{\overline{M};p,q}$, i.e. $A = A' \cup \{w\}$. Then $|A| = |D|$ and

$$|\partial_e A| = |\partial_e A'| + (4 - q)$$

and using Equation (4.5) and $q \geq 4$ we obtain

$$|\partial_e A| \leq |\partial_e D| + 6 - 2q < |\partial_e D|.$$

However, if $q = 3$, moving one vertex will not be enough to get a set with strictly smaller perimeter, we instead have to fill the whole empty strip with vertices from the last layer. Note that

$$|\partial_e B_{D,Min}| = |\partial_e D| - |S_e| - 3.$$

Call S'_o an occupied strip from $L_{\overline{M}}$ with cardinality $|S'_o| = |S_e|$ and order its vertices, such that the first vertex $w \in S'_o \cap E_{\overline{M};p,q}$. Fill the empty strip S_e. Denote this new set by A, then

$$|\partial_e A| = |\partial_e A'| + |S'_o(I)| - |S'_o(E)| + 2$$

and therefore $|\partial_e A| = |\partial_e D| - 2|S'_o(E)| - 1 < |\partial_e D|$.

`CASE (III)` For the general case we have $s_e > 1$ empty strips in the annulus $B_{D,Min} \setminus B_{D,max}$, denote them by $S_{e,1}, \ldots, S_{e,s_e}$ where s_e. We will distinguish $q \geq 4$ and $q = 3$. For examples, see Figures 4.3, resp. 4.4. We will first assume that there exists a strip $S_{e,j}$ which is in some layer L_k with $k < \overline{M}$ such that the vertices in the layers $L_{k-1} \cup L_{k+1}$ connected with $S_{e,j}$ are occupied.

Let us first consider $q \geq 4$. Consider any strip $S_{e,j} \ni v$. If we fill the empty vertex v, then

$$|\partial_e D \cup \{v\}| \leq |\partial_e D| + (2 - q), \tag{4.6}$$

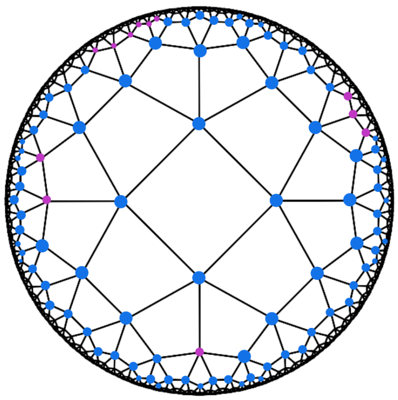

Figure 4.3: Example for a set D (blue sites) for $p = 4$ and $q \geq 4$. We highlighted the empty sites by pink vertices. We have two empty strips $S_{e,1}, S_{e,2}$ in L_1 and two empty strips $S_{e,3}, S_{e,4}$ in L_2.

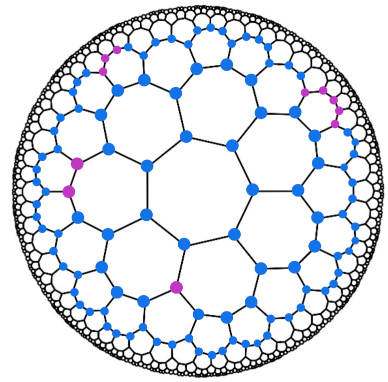

Figure 4.4: Example for a set D (blue sites) for $p = 7$ and $q = 3$. We highlighted the empty sites by pink vertices. We have two empty strips $S_{e,1}, S_{e,2}$ in L_1 and two empty strips $S_{e,3}, S_{e,4}$ in L_2.

see also Figure 4.3. If there exists a vertex $w \in E_{\overline{M};p,q} \cap \mathcal{N}_o$, then define $A \coloneqq (D \cup \{v\}) \setminus \{w\}$. The perimeter satisfies $|\partial_e A| = |\partial_e D \cup \{v\}| + (4 - q)$.

By Equation (4.6), we obtain the claim since $|\partial_e A| \leq |\partial_e D| + 6 - 2q < |\partial_e D|$. Otherwise if all occupied vertices $w \in L_{\overline{M}} \cap \mathcal{N}_o$ are in $I_{\overline{M};p,q}$, we there exists a vertex in $w' \in E_{\overline{M}-1;p,q} \cap \mathcal{N}_o$. Analogously we obtain for $A \coloneqq (D \cup \{v\}) \setminus \{w'\}$ that the perimeter can be bounded by

$$|\partial_e A| \leq |\partial_e D| + 4 - 2q < |\partial_e D|.$$

In the remaining argument let us assume that $q = 3$. We have to distinguish between the case that in $L_{\overline{M}} \cap \mathcal{N}_o$ there are enough vertices to fill $S_{e,j}$ or not. Let us first consider the case that $|L_{\overline{M}} \cap \mathcal{N}_o| \geq |S_{e,j}|$. Let $A' = D \cup S_{e,j}$, then the perimeter is equal to $|\partial_e A'| = |\partial_e D| - |S_{e,j}| - 3$. If the whole last layer is occupied, i.e. when $L_{\overline{M}} \cap \mathcal{N}_o = L_{\overline{M}}$, then define (and order) a strip in the last layer $S'_o = \{v_1, \ldots, v_K\}$ for $K = |S_{e,j}|$ such that $v_1 \in E_{\overline{M};p,3}$. Set $A = (D \cup S_{e,j}) \setminus S'_o$ and by a direct computation we get that $|\partial_e A| = |\partial_e D \cup S_{e,j}| + |S'_o(I)| - |S'_o(E)| + 2$ and therefore

$$|\partial_e A| \leq |\partial_e D| - 2|S'_o(E)| - 1 < |\partial_e D|.$$

Otherwise if there are empty vertices in $L_{\overline{M}}$, then define the strip $S'_o \subset L_{\overline{M}}$ with v_1 next to an empty vertex $w \in L_{\overline{M}} \cap \mathcal{N}_e$. Then we obtain for the perimeter, setting $A = (D \cup S_{e,j}) \setminus S'_o$, then

$$|\partial_e A| \leq |\partial_e D \cup S_{e,j}| + |S'_o(I)| - |S'_o(E)| \leq |\partial_e D| - 2|S'_o(E)| - 3 < |\partial_e D|.$$

For examples of the sets D, resp. A, see Figures 4.5 resp. 4.6.

Let us now focus on the case that $|L_{\overline{M}} \cap \mathcal{N}_o| < |S_{e,j}|$, hence there are less occupied vertices than $|S_{e,j}|$. Call $h \coloneqq |L_{\overline{M}} \cap \mathcal{N}_o|$ and fill h vertices $\{v_1, \ldots, v_h\}$ in $S_{e,j}$. Then

$$|\partial_e D \cup \{v_1, \ldots, v_h\}| = |\partial_e D| - h.$$

Then we define A from the previous set by removing all vertices from the last layer, i.e. $A \coloneqq (D \cup \{v_1, \ldots, v_h) \setminus (L_{\overline{M}} \cap \mathcal{N}_o)$. Then analogously

$$|\partial_e A| \leq |\partial_e D| - 2|\mathcal{N}_o \cap E_{\overline{M};p,3}| - 3 < |\partial_e D|.$$

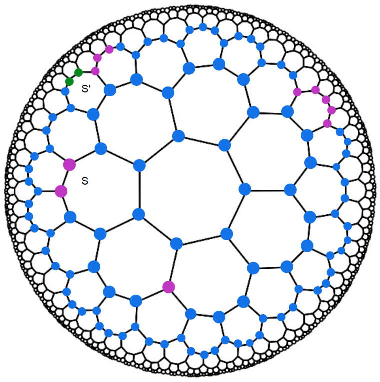

Figure 4.5: Example of a set D. We highlighted an empty strip S and a strip S' with the same cardinality with first vertex in $E_{2;7,3}$.

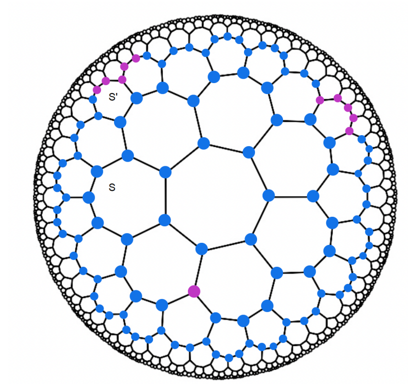

Figure 4.6: Example of a constructed set A obtained from the set D in Figure 4.5. We filled the empty strip S with vertices from S' with the same cardinality.

Assume now that a strip $S_{e,j}$ such that the occupied vertices in the layers $L_{k-1} \cup L_{k+1}$ are connected with $S_{e,j}$, does not exist. Recall that we have s_e empty (resp. occupied) strips in the annulus $B_{D,Min} \setminus B_{D,max}$. Denote by $S_{o,1}, \ldots, S_{o,s_e}$ the occupied strips in layers L_j where $j \in \{\overline{m}+1, \ldots, \overline{M}\}$. Recall that $\bigcup_{i=1}^{s_e} S_{o,i} = \mathcal{N}_o$. Then the number of occupied vertices in $\bigcup_{j=\overline{m}+1}^{\overline{M}} I_{j;p,3}$ satisfies

$$t := \left| \mathcal{N}_o \cap \left(\bigcup_{j=\overline{m}+1}^{\overline{M}} I_{j;p,3} \right) \right| < o_{max} + s_e - 1,$$

since by assumption, $D \notin \mathcal{M}_N$. Then for $D \setminus \mathcal{N}_o = B_{D,max}$ we have that

$$\begin{aligned} |\partial_e B_{D,max}| &\leq |\partial_e D| + \sum_{j=1}^{s_e} \big(|S_{o,j}(I)| - 1\big) - \sum_{j=1}^{s_e} \big(|S_{o,j}(E)| + 1\big) \\ &= |\partial_e D| + (t - s_e) - (|\mathcal{N}_o| - t + s_e). \end{aligned} \tag{4.7}$$

If $|\mathcal{N}_o| \leq |L_{\overline{m}+1}|$, we define $A := B_{D,max} \cup S'$ where $|S'| = |\mathcal{N}_o|$ and such that S' contains o_{max} vertices in $I_{\overline{m}+1;p,3}$. Otherwise there exists $k \in \{\overline{m}+1, \ldots, \overline{M}-1\}$ such that

$$\sum_{j=\overline{m}+1}^{k} |L_j| < |\mathcal{N}_o| \leq \sum_{j=\overline{m}+1}^{k+1} |L_j|.$$

In this case we define A by $A := B_{D,max} \cup \bigcup_{j=1}^{k} L_j \cup S'$, where S' is a strip of length $|\mathcal{N}_o| - \sum_{j=\overline{m}+1}^{k} |L_j|$ and such that S' contains the maximal number of vertices in $I_{k+1;p,3}$. For an example see Figure 4.7. Then using Equation (4.7)

$$|\partial_e A| \leq |\partial_e D| + (t - s_e - o_{max} + 1) < |\partial_e D| \tag{4.8}$$

since $t < o_{max} + s_e - 1$.

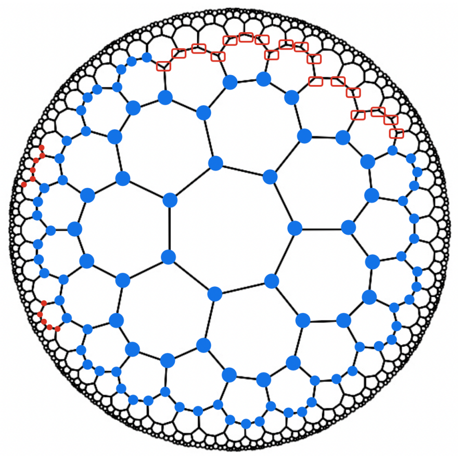

Figure 4.7: Example of a set D with two stripes of red occupied vertices, $S_{o,1}, S_{o,2}$ in the last layer and one empty stripe of length 18 in the previous layer which will be occupied by $|S_{o,1} \cup S_{o,2}| = 11$ many vertices.

4.2 Proof of Theorem 3.2

Remark that for a fixed $N \in \mathbb{N}$ and let $Q \in \mathcal{S}_N$, $\mathscr{M} \in \mathcal{M}_N$. By Theorem 3.1 we have that

$$\frac{|\partial_e Q|}{|Q|} \geq \frac{|\partial_e \mathscr{M}|}{|\mathscr{M}|} \geq \inf\left\{\frac{|\partial_e A|}{|A|} : 0 < |A| < \infty\right\}.$$

On the other hand by Theorem 4.1 in [6] we have that

$$i_e(\mathcal{L}_{p,q}) = \inf\left\{\frac{|\partial_e A|}{|A|} : 0 < |A| < \infty\right\} = (q-2)\sqrt{1 - \frac{4}{(p-2)(q-2)}}.$$

Thus, it follows trivially that

$$\lim_{N\to\infty} \frac{|\partial_e \mathscr{M}|}{|\mathscr{M}|} \geq i_e(\mathcal{L}_{p,q}). \tag{4.9}$$

The claim of the result will follow from Lemmas 4.2 and 4.3 which we will prove below.

Lemma 4.2. *We have that*

$$\lim_{n\to\infty} \frac{|\partial_e B_n(\mathbf{o})|}{|B_n(\mathbf{o})|} = i_e(\mathcal{L}_{p,q}).$$

Lemma 4.3. *For $\mathscr{M} \in \mathcal{M}_N$ we have that*

$$\lim_{|\mathscr{M}|\to\infty} \frac{|\partial_e \mathscr{M}|}{|\mathscr{M}|} \leq i_e(\mathcal{L}_{p,q}). \tag{4.10}$$

4.2.1 Proof of Lemma 4.2

Observe that the perimeter of a ball of radius n is equal to (depending on $p \geq 4$ resp. $p = 3$)

$$|\partial_e B_n(\mathbf{o})| = |I_{n+1;p,q}|, \quad \text{resp.} \quad |\partial_e B_n(\mathbf{o})| = 2|I''_{n+1;3,q}| + |I'_{n+1;3,q}|,$$

so that the ratio becomes

$$\frac{|\partial_e B_n(\mathbf{o})|}{|B_n(\mathbf{o})|} = \begin{cases} \frac{|I_{n+1;p,q}|}{\sum_{k=0}^{n}\left(|I_{k;p,q}|+|E_{k;p,q}|\right)} & \text{if } p \geq 4, \\ \\ \frac{2|I''_{n+1;3,q}|+|I'_{n;p,q}|}{3+\sum_{k=1}^{n}\left(|I'_{k;3,q}|+|I''_{k;3,q}|\right)} & \text{if } p = 3. \end{cases}$$

From [13] we have the following recursion relation for $I_{n;p,q}, E_{n;p,q}$, see also [13, Equations (2.14) and (2.16)], for $p \geq 4$ and $n \geq 0$

$$\begin{pmatrix} I_{n+1;p,q} \\ E_{n+1;p,q} \end{pmatrix} = T_1 \begin{pmatrix} I_{n;p,q} \\ E_{n;p,q} \end{pmatrix} \tag{4.11}$$

where $(I_{0;p,q}, E_{0;p,q}) = (0, p)$ and

$$T_1 = \begin{pmatrix} q-3 & q-2 \\ 8-3p-3q+pq & 5-2p-3q+pq \end{pmatrix} \tag{4.12}$$

resp. for $p = 3$, $n \geq 1$

$$\begin{pmatrix} I_{n+1;3,q} \\ E_{n+1;3,q} \end{pmatrix} = T_2 \begin{pmatrix} I_{n;3,q} \\ E_{n;3,q} \end{pmatrix} \tag{4.13}$$

where $(I_{0;3,q}, E_{0;3,q}) = (3, 3(q-4))$ and

$$T_2 = \begin{pmatrix} 1 & 1 \\ q-6 & q-5 \end{pmatrix}. \tag{4.14}$$

We will distinguish between $p \geq 4$ and $p = 3$. Let us consider the first case.

`CASE (I)` Due to the recursion relation, we have that

$$|I_{n+1;p,q}| = (1,0) T_1^{n+1} \begin{pmatrix} 0 \\ p \end{pmatrix}.$$

Diagonalizing T_1 we obtain that

$$T_1^n = (e_+, e_-) \begin{pmatrix} \lambda_+^n & 0 \\ 0 & \lambda_-^n \end{pmatrix} (e_+, e_-)^{-1},$$

where the eigenvalues are equal to

$$\begin{aligned} \lambda_\pm &= \frac{1}{2}\left(2 + p(q-2) - 2q \pm \sqrt{(p-2)(q-2)(q(p-2)-2p)}\right) \\ &= \left(\frac{1}{2}(p-2)(q-2) - 1\right)\left(1 \pm \sqrt{1 - \frac{4}{((p-2)(q-2)-2)^2}}\right), \end{aligned} \tag{4.15}$$

and the corresponding eigenvectors

$$e_+ = \left(\frac{2}{p-4+\frac{\sqrt{(p-2)(p(q-2)-2q)}}{\sqrt{q-2}}}, 1 \right)^T, \quad \text{resp. } e_- = \left(\frac{2}{p-4-\frac{\sqrt{(p-2)(p(q-2)-2q)}}{\sqrt{q-2}}}, 1 \right)^T.$$

Therefore,

$$|I_{n+1;p,q}| = \frac{p\sqrt{q-2}\left(\lambda_+^{n+1} - \lambda_-^{n+1}\right)}{\sqrt{(p-2)(p(q-2)-2q)}} \tag{4.16}$$

resp.

$$\begin{aligned}\sum_{k=0}^{n} \left(|I_{k;p,q}| + |E_{k;p,q}|\right) &= \sum_{k=0}^{n} (1,1) T_1^k \binom{0}{p} \\ &= \frac{p}{2\sqrt{(p-2)(p(q-2)-2q)}} \sum_{k=0}^{n} \left(a_- \lambda_-^k + a_+ \lambda_+^k\right)\end{aligned}$$

with

$$\begin{aligned} a_\pm &= (\sqrt{(p-2)(p(q-2)-2q)} \mp 2\sqrt{q-2} \pm p\sqrt{q-2}) \\ &= (p-2)\sqrt{q-2}\left(\pm 1 + \sqrt{1 - \frac{4}{(p-2)(q-2)}}\right). \end{aligned} \tag{4.17}$$

Then we can conclude that

$$\begin{aligned} \lim_{n\to\infty} \frac{|\partial_e(B_n(\mathbf{o})|}{|B_n(\mathbf{o})|} &= \lim_{n\to\infty} 2\sqrt{q-2} \frac{\left(\lambda_+^{n+1} - \lambda_-^{n+1}\right)}{\sum_{k=0}^{n}\left(a_-\lambda_-^k + a_+\lambda_+^k\right)} = \lim_{n\to\infty} 2\sqrt{q-2} \frac{\left(\lambda_+^{n+1} - \lambda_-^{n+1}\right)}{a_- \frac{1-\lambda_-^{n+1}}{1-\lambda_-} + a_+ \frac{1-\lambda_+^{n+1}}{1-\lambda_+}} \\ &= \lim_{n\to\infty} 2\sqrt{q-2} \frac{1 - \left(\frac{\lambda_-}{\lambda_+}\right)^{n+1}}{\frac{a_-}{1-\lambda_-}\frac{1-\lambda_-^{n+1}}{\lambda_+^{n+1}} + \frac{a_+}{1-\lambda_+}\frac{1-\lambda_+^{n+1}}{\lambda_+^{n+1}}} = 2\sqrt{q-2}\frac{\lambda_+ - 1}{a_+} \end{aligned} \tag{4.18}$$

since $\left(\frac{\lambda_-}{\lambda_+}\right)^n \to 0$ for $n \to \infty$. Moreover, by a direct computation we can compute that

$$i_e(\mathcal{L}_{p,q}) = 2\sqrt{q-2}\frac{\lambda_+ - 1}{a_+},$$

which was defined in Equation (3.1).

`CASE (II)` We diagonalize T_2 by computing its eigenvalues reps. eigenvectors:

$$\lambda_\pm = \frac{1}{2}\left(q - 4 \pm \sqrt{(q-6)(q-2)}\right) \tag{4.19}$$

resp.

$$e_+ = \left(\frac{6-q+\sqrt{12-8q+q^2}}{2(q-6)}, 1\right)^T, \quad \text{resp. } e_- = \left(\frac{6-q-\sqrt{12-8q+q^2}}{2(q-6)}, 1\right)^T. \tag{4.20}$$

Analogously to before we write for the fraction

$$\frac{|\partial_e B_n(\mathbf{o})|}{|B_n(\mathbf{o})|} = \frac{2(1,0)T_2^n \begin{pmatrix} 3 \\ 3(q-4) \end{pmatrix} + (0,1)T_2^n \begin{pmatrix} 3 \\ 3(q-4) \end{pmatrix}}{3 + \sum_{k=0}^{n-1}(1,1)T_2^k \begin{pmatrix} 3 \\ 3(q-4) \end{pmatrix}} \tag{4.21}$$

which reduces to

$$|I''_{n+1;3,q}| = \frac{3\left(\left(\sqrt{(q-6)(q-2)} - (q-2)\right)\lambda_-^n + \left(\sqrt{(q-6)(q-2)} + q - 2\right)\lambda_+^n\right)}{2\sqrt{(q-6)(q-2)}},$$

and

$$2|I''_{n+1;3,q}| + |I'_{n+1;3,q}| = \frac{3(q-2)\left(\left(4-q+\sqrt{(q-6)(q-2)}\right)\lambda_-^n + \left(q-4+\sqrt{(q-6)(q-2)}\right)\lambda_+^n\right)}{2\sqrt{(q-6)(q-2)}}$$

resp.

$$|B_n(\mathbf{o})| = 3 + \sum_{k=0}^{n-1}\left(a_+\lambda_+^k + a_-\lambda_-^k\right) = 3 + a_+\frac{1-\lambda_+^n}{1-\lambda_+} + a_-\frac{1-\lambda_-^n}{1-\lambda_-}, \tag{4.22}$$

where

$$a_\pm = \frac{3}{2}\left(q - 3 \pm \frac{(q-5)(q-2)}{\sqrt{(q-6)(q-2)}}\right).$$

By a direct computation, we obtain the claim

$$\begin{aligned} \lim_{n\to\infty}\frac{|\partial_e B_n(\mathbf{o})|}{|B_n(\mathbf{o})|} &= \frac{3}{2}\frac{q-2}{\sqrt{(q-6)(q-2)}}\left(q-4+\sqrt{(q-6)(q-2)}\right)\frac{\lambda_+-1}{a_+} \\ &= 3\sqrt{\frac{q-2}{q-6}}\frac{\lambda_+(\lambda_+-1)}{a_+} = i_e(\mathcal{L}_{3,q}) \end{aligned} \tag{4.23}$$

since $\left(\frac{\lambda_-}{\lambda_+}\right)^n \to 0$ for $n \to \infty$.

4.2.2 Proof of Lemma 4.3

In the following let us give an exact formula for the perimeter of $\mathscr{M}$. If N is such that $\mathcal{N}_o \neq \emptyset$, i.e. $\mathscr{M} \not\equiv B_n(\mathbf{o})$, then

$$|\partial_e\mathscr{M}| = \begin{cases} |I_{n+1;p,q}| + (q-4)(|\mathcal{N}_o \cap I_{n+1;p,q}| - 1) + (q-2)(|\mathcal{N}_o \cap E_{n+1;p,q}| + 1) & \text{if } p \geq 4, \\ 2|I''_{n+1;3,q}| + |I'_{n+1;3,q}| + (q-6)(|\mathcal{N}_o \cap I''_{n+1;3,q}| - 1) + (q-4)(|\mathcal{N}_o \cap I'_{n+1;3,q}| + 1) & \text{if } p = 3. \end{cases} \tag{4.24}$$

For an example see Figure 4.8.

We distinguish two cases: Case (I) $p \geq 4$, and Case (II) $p = 3$.

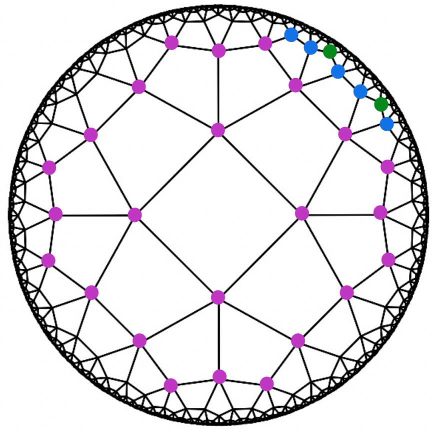

Figure 4.8: Example of a set $\mathscr{M}$ satisfying the relation (4.24) for $p=4$, $q=5$. We have that $|\mathcal{N}_o|=7$, $|\mathcal{N}_o\cap I_{2;4,5}|=5$, $|\mathcal{N}_o\cap E_{2;4,5}|=2$, $|I_{2;4,5}|=48$ and $|\partial_e\mathscr{M}|=61=48+1\cdot(5-1)+3\cdot(2+1)$

`CASE (I)` We note that $|\mathcal{N}_o\cap I_{n+1;p,q}|\geq \frac{|I_{n+1;p,q}|}{|L_{n+1}|}|\mathcal{N}_o|$, see Remark 4.4 for details. By using Equation (4.24), we can bound the ratio by

$$\begin{aligned}\frac{|\partial_e\mathscr{M}|}{|\mathscr{M}|}&=\frac{|I_{n+1;p,q}|+(q-4)(|\mathcal{N}_o\cap I_{n+1;p,q}|-1)+(q-2)(|\mathcal{N}_o\cap E_{n+1;p,q}|+1)}{\left|\bigcup_{j=0}^n L_j\right|+|\mathcal{N}_o|}\\&=\frac{|I_{n+1;p,q}|+(q-2)|\mathcal{N}_o|-2|\mathcal{N}_o\cap I_{n+1;p,q}|+2}{\left|\bigcup_{j=0}^n L_j\right|+|\mathcal{N}_o|}\\&\leq\frac{|I_{n+1;p,q}|+\left(q-2-2\frac{|I_{n+1}|}{|L_{n+1}|}\right)|\mathcal{N}_o|+2}{\left|\bigcup_{j=0}^n L_j\right|+|\mathcal{N}_o|}.\end{aligned}\tag{4.25}$$

Since $|\bigcup_{j=0}^n L_j|>0$ and $\left(q-2-2\frac{|I_{n+1}|}{|L_{n+1}|}\right)>1$, see Remark 4.5 for details, we have an increasing function of $|\mathcal{N}_o|<|L_{n+1}|$ and we obtain

$$\begin{aligned}\frac{|\partial_e\mathscr{M}|}{|\mathscr{M}|}&<\frac{|I_{n+1;p,q}|+\left(q-2-2\frac{|I_{n+1}|}{|L_{n+1}|}\right)|L_{n+1}|+2}{\left|\bigcup_{j=0}^n L_j\right|+|L_{n+1}|}\\&=\frac{|I_{n+1;p,q}|+(q-2)\left(|I_{n+1;p,q}|+|E_{n+1;p,q}|\right)-2|I_{n+1;p,q}|+2}{\left|\bigcup_{j=0}^{n+1} L_j\right|}\\&=\frac{\overbrace{(q-3)|I_{n+1;p,q}|+(q-2)|E_{n+1;p,q}|}^{|I_{n+2;p,q}|}+2}{\left|\bigcup_{j=0}^{n+1} L_j\right|}\\&=\frac{|I_{n+2;p,q}|+2}{\left|\bigcup_{j=0}^{n+1} L_j\right|}\to i(\mathcal{L}_{p,q}).\end{aligned}\tag{4.26}$$

`CASE (II)` First, we note that if $p = 3$ then $q \geq 7$ by the assumption $1/p + 1/q < 1/2$. Moreover, we observe that $|\mathcal{N}_o \cap I''_{n+1;p,q}| \geq \frac{|I''_{n+1;p,q}|}{|L_{n+1}|}|\mathcal{N}_o|$, thus

$$\begin{aligned}\frac{|\partial_e \mathscr{M}|}{|\mathscr{M}|} &= \frac{2|I''_{n+1;p,q}| + |I'_{n+1;p,q}| + (q-6)(|\mathcal{N}_o \cap I''_{n+1;p,q}| - 1) + (q-4)(|\mathcal{N}_o \cap I'_{n+1;p,q}| + 1)}{|\bigcup_{j=0}^{n} L_j| + |\mathcal{N}_o|} \\ &= \frac{2|I''_{n+1;p,q}| + |I'_{n+1;p,q}| + (q-4)|\mathcal{N}_o| - 2|\mathcal{N}_o \cap I''_{n+1;p,q}| + 2}{|\bigcup_{j=0}^{n} L_j| + |\mathcal{N}_o|} \\ &\leq \frac{2|I''_{n+1;p,q}| + |I'_{n+1;p,q}| + \left(q - 4 - 2\frac{|I''_{n+1;p,q}|}{|L_{n+1}|}\right)|\mathcal{N}_o| + 2}{|\bigcup_{j=0}^{n} L_j| + |\mathcal{N}_o|}. \end{aligned} \tag{4.27}$$

Since $|\bigcup_{j=0}^{n} L_j| > 0$ and $\left(q - 4 - 2\frac{|I''_{n+1;p,q}|}{|L_{n+1}|}\right) > 1$, see Remark 4.5 for details, we again have an increasing function of $|\mathcal{N}_o| < |L_{n+1}|$ and we obtain

$$\begin{aligned}\frac{|\partial_e \mathscr{M}|}{|\mathscr{M}|} &< \frac{|I'_{n+1;p,q}| + (q-4)|L_{n+1}| + 2}{|\bigcup_{j=0}^{n+1} L_j|} \\ &= \frac{(q-3)|I'_{n+1;p,q}| + (q-4)|I''_{n+1;p,q}| + 2}{|\bigcup_{j=0}^{n+1} L_j|} \\ &= \frac{\overbrace{(q-5)|I'_{n+1;p,q}| + (q-6)|I''_{n+1;p,q}|}^{I'_{n+2;p,q}} + \overbrace{2|I'_{n+1;p,q}| + 2|I''_{n+1;p,q}|}^{2I''_{n+2;p,q}} + 2}{|\bigcup_{j=0}^{n+1} L_j|} \\ &\to i(\mathcal{L}_{p,q}) \end{aligned} \tag{4.28}$$

Remark 4.4. By Definition 2.6, we recall that $|\mathcal{N}_o|$ contains the maximal number of vertices in $I_{n+1;p,q}$, i.e., $|\mathcal{N}_o \cap I_{n+1;p,q}|$ is maximal. For $j = 1, ..., |L_{n+1}|$, we define S_j the ordered strips L_{n+1} of length $|S_j| = |\mathcal{N}_o|$. Thus, we obtain

$$|I_{n+1;p,q}| = \sum_{j=1}^{|L_{n+1}|} \frac{|S_j \cap I_{n+1;p,q}|}{|S_j|} \leq |L_{n+1}| \max_{j \in \{1,...,|L_{n+1}|\}} \frac{|S_j \cap I_{n+1;p,q}|}{|S_j|} = |L_{n+1}| \frac{|\mathcal{N}_o \cap I_{n+1;p,q}|}{|\mathcal{N}_o|} \tag{4.29}$$

Remark 4.5. We prove that $q - 2 - 2\frac{|I_{n+1;p,q}|}{|L_{n+1}|} > 1$ and $q - 4 - 2\frac{|I''_{n+1;p,q}|}{|L_{n+1}|} > 1$. By definition of $I_{n+1;p,q}$ and recall that the eigenvalues $\lambda_\pm$ were defined in Equation (4.19) and the constants $a_\pm$ in Equation (4.17),

$$\begin{aligned} q - 2 - 2\frac{|I_{n+1;p,q}|}{|L_{n+1}|} &= q - 2 - 4\frac{\sqrt{q-2}(\lambda_+^{n+1} - \lambda_-^{n+1})}{a_+\lambda_+^{n+1} + a_-\lambda_-^{n+1}} \geq q - 2 - 4\frac{\sqrt{q-2}}{a_+} \\ &= q - 2 - \frac{4}{(p-2)\left(1 + \sqrt{1 - \frac{4}{(p-2)(q-2)}}\right)} > 1 \end{aligned} \tag{4.30}$$

where we used that the first function is decreasing in n, the second function is increasing in p and

q with $1/p + 1/q < 1/2$. Moreover,

$$g(q,n) := q - 4 - 2\frac{|I''_{n+1;p,q}|}{|L_{n+1}|} = q - 4 - \frac{3\left(\left(1 - \sqrt{\frac{q-2}{q-6}}\right)\lambda_-^{n+1} + \left(1 + \sqrt{\frac{q-2}{q-6}}\right)\lambda_+^{n+1}\right)}{a_+\lambda_+^{n+1} + a_-\lambda_-^{n+1}}$$

$$\geq q - 4 - \frac{3\left(\left(1 - \sqrt{\frac{q-2}{q-6}}\right)\lambda_- + \left(1 + \sqrt{\frac{q-2}{q-6}}\right)\lambda_+\right)}{a_+\lambda_+ + a_-\lambda_-} \geq g(7,1) > 1 \tag{4.31}$$

where we used that the first function is increasing in n and the second function is increasing in q, where $q \geq 7$.

4.3 Proof of Proposition 3.3

We know that the tile $\mathcal{T} := L_0$ is a generating set for $\mathcal{G}_{p,q}$. The word norm induced by $\mathcal{T}$ in $\mathcal{G}_{p,q}$ is the minimal length g of a word respectively to the Fuchsian group in $\mathcal{T}$. Let $g(z)$ be the growth series,

$$g(z) = a_0 + a_1 z + a_2 z^2 + \dots$$

where a_n is the number of elements in $\mathcal{G}_{p,q}$ with word norm exactly n. Recall that we are considering our hyperbolic lattice starting from a tile and not from a vertex. The coefficients a_n of the growth series will depend on p, we have that for $n \geq 0$

$$a_n^{(p)} = (1,1)T_1^n \begin{pmatrix} 0 \\ p \end{pmatrix}$$

for $p \geq 4$ and for $p = 3$ and $n \geq 1$

$$a_n^{(3)} = 3 + (1,1)T_2^{n-1} \begin{pmatrix} 3 \\ 3(q-4) \end{pmatrix},$$

where the matrices T_1, T_2 were defined in Equations (4.12) resp. (4.14). In the first case we write

$$g(z) = p + (1,1)\left(\sum_{n=1}^{\infty} z^{n-1} T_1^{n-1}\right) z\, v \tag{4.32}$$

where the vector v is equal to $v = T_1 \binom{0}{p}$.

We note that the sum in this *formal* function g is the Maclaurin series of the following rational function

$$f_1(z) = p + (1,1)(\mathcal{I}d - zT_1)^{-1} z\, v,$$

where $\mathcal{I}d$ is the 2×2 identity matrix. By a direct computation, we obtain that

$$f_1(z) = \frac{p(1+z)}{z^2 + (2q + 2p - pq - 2)z + 1}. \tag{4.33}$$

By applying [12, Theorem 1], we find that the Euler characteristic is equal to

$$\mathcal{X}(\mathcal{G}_{pq}) = \frac{1}{f_1(1)} = \frac{2q + 2p - pq}{2p}. \tag{4.34}$$

In the following, we compute the Euler characteristic for $p = 3$. The growth series $h(z)$ is equal to

$$h(z) = 3 + \sum_{n=0}^{\infty}(1,1)(T_2 z)^n \begin{pmatrix} 3 \\ 3(q-4) \end{pmatrix} = 3(q-2) + (1,1)\left(\sum_{n=1}^{\infty} z^{n-1} T_2^{n-1}\right) z\, u \tag{4.35}$$

where the vector u is equal to $u = T_2 \binom{3}{3(q-4)}$. Then

$$f_2(z) = 3(q-2) + (1,1)(\mathcal{I}d - zT_2)^{-1} z\, u,$$

or equivalently

$$f_2(z) = 3 + \frac{3(q-3-z)}{1+z(4-q+z)}. \tag{4.36}$$

We apply again [12, Theorem 1] and obtain that

$$\mathcal{X}(\mathcal{G}_{3,q}) = \frac{6-q}{6} \tag{4.37}$$

which coincides with the Euler characteristic we have found before in Equation (4.34) setting $p = 3$.

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References

[1] M. Aizenman, *Translation invariance and instability of phase coexistence in the two dimensional Ising system*, Communications in Mathematical Physics, 73 (1980), pp. 83–94.

[2] I. Boettcher, A. V. Gorshkov, A. J. Kollár, J. Maciejko, S. Rayan, and R. Thomale, *Crystallography of hyperbolic lattices*, Physical Review. B, 105 (2022).

[3] A. Chen, H. Brand, T. Helbig, T. Hofmann, S. Imhof, A. Fritzsche, T. Kiessling, A. Stegmaier, L. K. Upreti, T. Neupert, et al., *Hyperbolic matter in electrical circuits with tunable complex phases*, Nature Communications, 14 (2023), p. 622.

[4] M. D'Achille, L. Coquille, and A. Le Ny, *Extremal Ising Gibbs States on Hyperbolic Lattices*, (2024^+). To appear.

[5] G. R. Grimmett and C. M. Newman, *Percolation in $\infty + 1$ dimensions*, Claremond Press, Oxford, 1990, pp. 219–240.

[6] O. Häggström, J. Jonasson, and R. Lyons, *Explicit isoperimetric constants and phase transitions in the random-cluster model*, The Annals of Probability, 30 (2002), pp. 443–473.

[7] Y. Higuchi, *On the absence of non translation invariant Gibbs states for the two dimensional Ising model*, in Colloquia Math. Sociatatis Janos Bolyai, vol. 27, Random fields, 1979, pp. 517–534.

[8] A. Kollár, M. Fitzpatrick, and A. Houck, *Hyperbolic lattices in circuit quantum electrodynamics*, Nature, 571 (2019), pp. 45–50.

[9] M. l Lawniczak, P. Kurasov, S. Bauch, M. Białous, and L. Sirko, *Euler Characteristic of Graphs and Networks*, Acta Physica Polonica A, (2021).

[10] S. Mertens and C. Moore, *Percolation thresholds in hyperbolic lattices*, Physical Review E, 96 (2017), p. 042116.

[11] J. F. Moran, *The growth rate and balance of homogeneous tilings in the hyperbolic plane*, Discrete Mathematics, 173 (1997), pp. 151–186.

[12] S. P. Plotnick and W. J. Floyd, *Growth functions on fuchsian groups and the euler characteristic*, Inventiones Mathematicae, 88 (1987), pp. 1–29.

[13] R. Rietman, B. Nienhuis, and J. Oitmaa, *The Ising model on hyperlattices*, Journal of Physics A: Mathematical and General, 25 (1992), p. 6577.

[14] M. Schrauth, Y. Thurn, F. Goth, J. S. Portela, D. Herdt, and F. Dusel, *The hypertiling project*, https://doi.org/10.5281/zenodo.7559393, (2023).

[15] C. Series and Y. G. Sinai, *Ising models on the Lobachevsky plane*, Communications in Mathematical Physics, 128 (1990), pp. 63–76.

[16] A. Smith and V. M. Zavala, *The Euler characteristic: A general topological descriptor for complex data*, Computers and Chemical Engineering, 154 (2021), p. 107463.

[17] S. Yu, X. Piao, and N. Park, *Topological Hyperbolic Lattices*, Phys. Rev. Lett., 125 (2020), p. 053901.