Exercise session 1*

for the lecture "The Assignment Problem from von Neumann to the Hungarian algorithm"

Nesin Mathematics Village, Turkey, 28/07/2024–4/08/2024

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Nota bene: The exercises are ranked in order of increasing difficulty, from direct application of classroom material (✱) to research-level problems (✱✱✱✱✱).

Exercise 1 (✱) Using the Hungarian algorithm, find an optimal assignment and the optimal cost for the following cost matrices:

(a)

$$c_1 = \begin{pmatrix} 3 & 5 & 2 & 3 \\ 7 & 1 & 2 & 9 \\ 2 & 4 & 4 & 1 \\ 3 & 5 & 2 & 7 \end{pmatrix}$$

(b)

$$c_2 = \begin{pmatrix} -2 & 1 & 4 & 2 \\ 1 & 4 & 1 & -2 \\ 3 & -2 & 0 & 1 \\ 5 & 2 & 1 & 4 \end{pmatrix}$$

(c)

$$c_3 = \begin{pmatrix} 6 & 1 & 4 & 1 & 8 \\ 4 & 5 & 9 & 4 & 3 \\ 6 & 1 & 5 & 6 & 4 \\ 6 & 2 & 3 & 1 & 4 \\ 1 & 6 & 5 & 8 & 0 \end{pmatrix}$$

(d)

$$c_4 = \begin{pmatrix} 7 & 5 & 8 & 10 & 2 \\ 11 & 6 & 13 & 11 & 6 \\ 15 & 8 & 7 & 8 & 7 \\ 9 & 6 & 10 & 17 & 6 \\ 7 & 5 & 7 & 8 & 2 \end{pmatrix}$$

*Latest version (July 30, 2024) available electronically at: https://matteodachille.github.io/teaching

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Exercise 2 (✱✱) Let $(x_i)_{i=1}^n$ be a strictly increasing real sequence and $(y_i)_{i=1}^n$ a strictly decreasing real sequence. Consider the assignment problem for the following cost matrix:

$$c_{ij} \stackrel{\text{def}}{=} x_i y_j, \quad i,j = 1, \dots, n \, .$$

✎ Prove that $\pi_{\text{opt}} = (1 \cdots n)$, and hence $c_{\text{opt}} = \operatorname{Tr} c$.
– *Hint:* reason either by induction or by contradiction.

Exercise 3 (✱✱✱✱) (von Neumann's game) The exercise consists of two independent parts.

3.1 Let $(\alpha_i)_{i=1}^n$ be a sequence of positive reals and let $p = (p_i)_{i=1}^n$ be a probability distr. over $[n]$.
✎ The goal is to prove that a probability distr. over $[n]$ such that

$$p^* = \operatorname*{arg\,min}_{p:\, p \text{ is a proba over } [n]} \max_{i \in [n]} (\alpha_i p_i)$$

is given by

$$p_i^* = \frac{\mathrm{HM}(\alpha_1, \dots, \alpha_n)}{\alpha_i \cdot n}, \qquad i = 1, \dots, n,$$

where $\mathrm{HM}(\alpha_1, \dots, \alpha_n) \stackrel{\text{def}}{=} \frac{n}{\sum_{i=1}^n \frac{1}{\alpha_i}}$ is the *harmonic mean* of $\alpha_1, \dots, \alpha_n$.

We will do it in two steps:

a) Prove that a p^* must satisfy $\alpha_1 p_1^* = \ldots = \alpha_n p_n^* = \Lambda$, where $\Lambda > 0$ is strictly positive.
– *Hint:* reason by contradiction.

b) Solve for p^* and Λ, and then comment.

3.2 Let $(\alpha_{i,j})_{i,j=1}^n$ be n^2 positive reals, let $\mathbf{B}_n$ be the Birkhhoff polytope of $n \times n$ doubly stochastic matrices and let $\mathbf{\Pi}_n$ be the set of probability distributions over $[n]^2$. The goal is to find a matrix $X^* \in \mathbf{\Pi}_n$ which satisfies the following:

$$X^* = \operatorname*{arg\,min}_{X \in \mathbf{\Pi}_n} \max_{\substack{r \in [n] \\ s \in [n]}} \left(\sum_{j=1}^n \alpha_{rj} X_{rj}, \sum_{i=1}^n \alpha_{is} X_{is} \right) \, .$$

We will do it in four steps:

a) Let $\Lambda > 0$ be the maximum value attained in the previous display (why can we do this?). Prove that

$$\zeta_{ij} \stackrel{\text{def}}{=} \frac{\alpha_{ij} X_{ij}^*}{\Lambda} \in \mathbf{B}_n \, .$$

b) Prove that ζ is a permutation matrix, for some permutation π_{opt} (to be determined).
– *Hint:* use the Birkhhoff–von Neumann theorem and the optimality requirement.

c) Deduce that $\sum_{j=1}^{n} \alpha_{ij} X_{ij}^* = \Lambda = \sum_{i=1}^{n} \alpha_{ij} X_{ij}^*$, and hence

$$\Lambda = \frac{1}{n} \mathrm{HM}(\alpha_{1\pi_{\mathrm{opt}}(1)}, \ldots, \alpha_{n\pi_{\mathrm{opt}}(n)}) \ .$$

d) Provide a cost matrix c depending on $(\alpha_{i,j})_{i,j=1}^{n}$ whose optimal assignment is π_{opt}.