Exercise session 2[*]

for the lecture "Adding Probability: Random Assignment Problems (RAPs)"

Nesin Mathematics Village, Turkey, 29/07/2024–4/08/2024

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Exercise 1 (✱✱) (Random Monge cost matrix) Let $\mathbf{X} = (X_i)_{i=1}^n$ and $\mathbf{Y} = (Y_i)_{i=1}^n$ be two iid samples of $U[0,1]$ random variables, $\mathbf{X} \perp\!\!\!\perp \mathbf{Y}$. Consider the RAP for the following matrix:

$$c_{ij} \stackrel{\text{def}}{=} X_i Y_j, \quad i, j = 1, \ldots, n \ .$$

✎ Prove that

$$\mathbb{E}[E_{\text{opt}}] = \frac{n(n+2)}{6(n+1)} \ .$$

Exercise 2 (✱✱✱) (Universal limit constant in the greedy strategy) Consider the RAP for the cost matrix $(c_{ij})_{i,j=1}^n$, where the c_{ij} are iid rvs of law X. Consider the n-th most smallest entries of c and denote by E_{greedy} the associated random cost, i.e.

$$E_{\text{greedy}} = \sum_{k=1}^{n} c_{(k)},$$

where $c_{(k)}$ is the k-th smallest entry of c (often called $k-$th *order statistics*).

✎ Prove that, for both $X = \text{Exp}(1)$ or $X = U[0,1]$,

$$E_{\text{greedy}} \xrightarrow[n\to\infty]{p} \frac{1}{2} \ .$$

[*]Available electronically at: https://matteodachille.github.io/teaching

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Exercise 3 (✱✱✱✱✱) (Recursive corner argument) Consider the RAP within the same setting of Exercise 2 and call E_{opt} its optimal cost. Consider the following auxiliary problem: Let $\left((c_{ij}^{(k)})_{i,j=1}^{n-k+1}\right)_{k=1}^{n}$ a sequence of n iid cost matrices of decreasing sizes, whose entries have law X. For $k = 1, \ldots, n$ consider the sequence of minima of each matrix by $C_k^* = \min c^{(k)}$ and define the following energy

$$E_{\text{rca}} \stackrel{\text{def}}{=} \sum_{k=1}^{n} C_k^* \, .$$

a) Prove that $E_{\text{rca}} \preceq E_{\text{opt}}$, where $\preceq$ denotes first order stochastic domination.

b) Prove that, for $X = U[0,1]$

$$\lim_{n\to\infty} \mathbb{E}[E_{\text{rca}}] = \mathrm{c} \in \left(1, \frac{\pi^2}{6}\right) \, .$$