Exercise session 3*

for the lecture "Adding Geometry: Euclidean Random Assignment Problems (ERAPs) and extensions,

featuring a crash course on point processes"

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Exercise 1 (✱✱) (Hole probability) Consider a Poisson Point Process (PPP) $\mathbf{X} = (X_i\,;\ i \geq 1)$ of points in the plane with intensity measure $\mathrm{Leb}_{\mathbb{R}^2}$. Let D_1 be the unit disk, C_1 the unit circle, and P_n a regular n-gon inscribed in C_1 for $n \geq 3$. Let $\mathcal{R}_n = D_1 \setminus P_n$ (see Fig. 1 for $n = 5$).

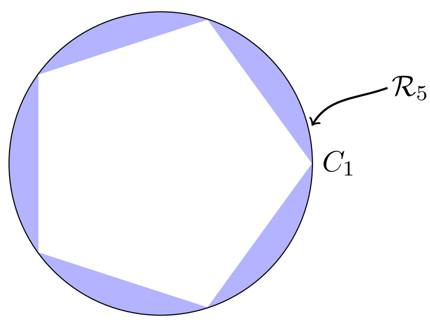

Fig. 1: The region $\mathcal{R}_5$ is shaded in blue.

✎ What is the smallest n s.t. the probability that no point of $\mathbf{X}$ falls into $\mathcal{R}_n$ is greater than $\frac{1}{2}$?

Exercise 2 (✱✱) (Quadratic ERAP on the unit interval) Consider the ERAP with uniform disorder over the unit interval $[0,1]$ with $p = 2$.

✎ Taking inspiration from Exercise 1 of Session 2, prove that

$$\mathbb{E}[\mathcal{H}_{\mathrm{opt}}] = \frac{1}{3}\frac{n}{n+1}\,. \tag{1}$$

*Latest version (August 3, 2024) available electronically at: https://matteodachille.github.io/teaching

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Exercise 3 (✱✱) (QUARTIC ERAP ON THE UNIT INTERVAL) Consider the ERAP with uniform disorder over the unit interval $[0,1]$ with $p = 4$.

✎ Taking inspiration from Exercise 2 above, prove that

$$\mathbb{E}[\mathcal{H}_{\text{opt}}] = \frac{2}{5}\frac{n}{(n+1)(n+2)} \,. \tag{2}$$

Exercise 4 (✱✱✱✱✱) (NICE FORMULAS) Consider $(E, \mathcal{D}) = ([0,1], |\cdot|)$, with disorder $\nu = \text{Unif}_{[0,1]}$. Caracciolo *et al.* [1] proved that, calling $B_{(k)}$ the k-th order statistics of $\mathcal{B}$ (and analogously for $\mathcal{R}$), for all integer $\ell \geq 1$ and $n \in \mathbb{N}$,

$$\mathbb{E}[|B_{(k)} - R_{(k)}|^\ell] = \frac{\Gamma^2(n+1)\Gamma(k+\frac{\ell}{2})\Gamma(n-k+1+\frac{\ell}{2})\Gamma(1+\ell)}{\Gamma(k)\Gamma(n-k+1)\Gamma(n+1+\frac{\ell}{2})\Gamma(n+1+\ell)\Gamma(1+\frac{\ell}{2})}, \; k = 1, \ldots, n \,. \tag{3}$$

For the choice of cost function $f = \mathcal{D}^p$, with $p > 1$, Equation (3) provides directly a closed formula for $\mathbb{E}[\mathcal{H}_{\text{opt}}]$, valid $\forall n \in \mathbb{N}$ and $\forall p > 1$, namely

$$\mathbb{E}[\mathcal{H}_{\text{opt}}] = \frac{\Gamma(1+p/2)}{p+1}\frac{\Gamma(n+1)}{\Gamma(n+1+p/2)}\, n \tag{4}$$

(remark that Equation (4) reduces to Equation (1) if $p = 2$ and to Equation (2) if $p = 4$).

✎ Are there other choices of $f = f(\mathcal{D})$ for which *Eq.* (3) provides a nice formula for $\mathbb{E}[\mathcal{H}_{\text{opt}}]$?

References

[1] CARACCIOLO, S., DI GIOACCHINO, A., MALATESTA, E. M., AND MOLINARI, L. G. Selberg integrals in 1D random Euclidean optimization problems. *Journal of Statistical Mechanics: Theory and Experiment 2019*, 6 (jun 2019), 063401.