# Exercise session 4[*]

for the lecture "The phase diagram of ERAPs in $d = 1$"

Nesin Mathematics Village, Turkey, 28/07/2024–4/08/2024

Matteo D'Achille[†]

**Exercise 1 (✱)** (Basic properties of the standard Brownian bridge) Recall the definition of the standard brownian Bridge $B_\tau$ by the following equality in law

$$B_\tau = W_\tau - \tau W_1, \quad \tau \in [0, 1] \,.$$

where $W_\tau$ denotes one dimensional Wiener process.

✎ Show the following:

a) Prove the following equality in law

$$\sum_{k=1}^{n} B_\tau^{(k)} \stackrel{\text{law}}{=} \sqrt{n}\, B_\tau$$

where $(B_\tau^{(k)})_{k=1}^{n}$ are $n$ independent standard Brownian bridges.

b) Given two times $t_1, t_2 \in [0, 1]$, $t_2 > t_1$, and two positions $x_1, x_2 \in \mathbb{R}$, provide an expression for the conditional probability density of $B_{t_2} = x_2$ given $B_{t_1} = x_1$.

**Exercise 2 (✱✱✱✱)** (ERAP and a functional of the Brownian bridge) Consider the standard Brownian Bridge $B_\tau$ on $[0, 1]$ defined in Exercise 1. Define the following functional

$$\Phi(B_\tau; \Lambda_p) \stackrel{\text{def}}{=} \int_0^1 |B_\tau - \Lambda_p|^p d\tau \,, \tag{1}$$

where $\Lambda_p$ is a real-valued random shift independent on $\tau$. The goal of the exercise is to study the functional 1 via a variational principle.

✎ Address the following three points:

a) Discuss $\exists$ and ! of extremizers $\Lambda_p^*$ depending on $p$.

b) Prove that $\forall p \geq 1$, any minimizer $\Lambda_p^*$ is centered. Provide a geometrical interpretation of $\Lambda_p^*$. What does $\Lambda_2^*$ represent? Relate it to the solution of Exercise 2, Session 3. Same for $\Lambda_1^*$.

c) Prove that $\frac{d}{dp}\mathbb{E}\left[(\Lambda_p^*)^2\right] > 0$ for $p \in [1, \infty)$.

[*]Latest version (August 3, 2024) available electronically at: https://matteodachille.github.io/teaching

[†]matteo.dachille@universite-paris-saclay.fr, solutions welcome!