Extremal Ising Gibbs States on Lobachevsky planes

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Abstract

We exhibit an uncountable family of extremal inhomogeneous Gibbs measures of the low temperature Ising model on regular tilings of the hyperbolic plane. These states arise as low temperature perturbations of local ground states having a sparse enough set of frustrated edges, the sparseness being measured in terms of the isoperimetric constant of the graph. This result is implied by an extension of the article [9] on regular trees to non-amenable graphs. We argue how we can deduce the extremality of an uncountable subset of the Series–Sinai states [30] at low temperature.

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1 Introduction

Since a pioneering work by Lund–Rasetti–Regge [24], the ferromagnetic n.n. Ising model on lattices which are naturally embedded in the hyperbolic plane $\mathbb{H}_2$ has attracted considerable interest in the physics literature. See also later the work by Rietman–Nienhuis–Oitmaa [28] or the more recent applied work in [6]. These works rose many interesting mathematical questions about the structure of Gibbs states on these lattices.

We recall that on the Euclidean lattice $\mathbb{Z}^2$ (corresponding to the tiling $\mathcal{L}_{4,4}$ below), a celebrated result due to Aizenmann [1] and Higuchi [19] states that any Gibbs measure writes as a convex combination of the pure phases μ^+ and μ^-. On $\mathbb{Z}^3$, the celebrated result of Dobrushin [11] exhibits a countable family of (extremal [12]) non translation-invariant states at low temperature, with a localised interface at a given height.

When the underlying graph is a d-regular tree, the Ising model possesses uncountably many such interface states in the low temperature regime $\beta > \beta_c(d)$, as it was proved by Higuchi in the late seventies [18] and by Bleher and Ganikhodaev [3] using a similar strategy afterwards. These states are extremal and non tree-automorphism invariant.
In 2008, Rozikov and Rakhmatullaev [29] exhibit the so-called "weakly periodic" Gibbs measures, which correspond to subgroups in the representation group of the Cayley tree. These states can be thought of as generalizations of Dobrushin states, but with many interfaces, possibly countably infinitely many, localised on a symmetric pattern.
In 2012, Gandolfo, Ruiz and Shlosman [14, 15], exhibit extremal inhomogeneous Gibbs measures arising as low temperature perturbations of ground states, which have a sparse enough set of frustrated bonds. These states in general do not possess any symmetries of the tree. The two last authors, together with Külske rigorously proved in [9] the statements of [14] and generalise them to a broader setting: non-compact state space gradient models, models without spin-symmetry, models in small random fields.

On hyperbolic lattices, Series and Sinai [30] exhibit an uncountable family of Gibbs states indexed by the geodesics of $\mathbb{H}_2$, see Figure 1.1 on the left for an example. The result of Series and

Sinai holds for the ferromagnetic *n.n.* Ising model defined on the vertices of the Cayley graph of any finitely generated co-compact group of isometries of $\mathbb{H}_2$, see [30, Theorem 1].
More recently, Gandolfo–Ruiz–Shlosman [15], building on Series–Sinai's paper, construct an uncountable family of inhomogeneous Gibbs states with multiple rigid interfaces called the *Mille-feuilles*.

In this paper we adapt the techniques of [9] to exhibit an uncountable family of infinite-volume Gibbs measures, whose existence and extremality is proven by means of adequate Peierls estimates and excess energy around local ground states.

We emphasize an important feature of Gibbs states of the Ising model on trees, which is the presence of a double transition: there exist a so-called "spin-glass" inverse temperature $\beta_{SG} \in (\beta_c, \infty)$ above which the free state (obtained as the infinite-volume Gibbs measure with free boundary conditions) is not extremal, and below which it is. This was shown in the sequence of works [18, 2, 4, 22, 21]. Note that the free state is never extremal on $\mathbb{Z}^d$ [5].

The extremal decomposition of the Ising free state on a tree is known to be particularly complex as it involves uncountably many Gibbs measures on which the extremal decomposition measure is supported, see the much more recent works of Gandolfo, Maes, Ruiz and Shlosman [13], and of Coquille, Külske, Le Ny [8]. These states have a broken translational symmetry, and show characteristics of a "glassy" behavior.

An analogous double transition occurs on hyperbolic lattices. Indeed, Wu [31] proved the existence of two inverse temperatures $0 < \beta_c < \beta_{\mathrm{SG}} < \infty$ such that: there exists a unique Gibbs state $(\mu^+ = \mu^-)$ at high temperatures $\beta < \beta_c$; in the intermediate temperature regime $\mu^+ \neq \mu^-$ and $\mu^{\mathrm{f}} \neq \frac{1}{2}\left(\mu^+ + \mu^-\right)$ for $\beta \in (\beta_c, \beta_{\mathrm{SG}})$; and at low temperatures $\mu^+ \neq \mu^-$ but $\mu^{\mathrm{f}} = \frac{1}{2}\left(\mu^+ + \mu^-\right)$ for $\beta > \beta_{\mathrm{SG}}$. The free state is expected to be extremal in the intermediate temperature regime.

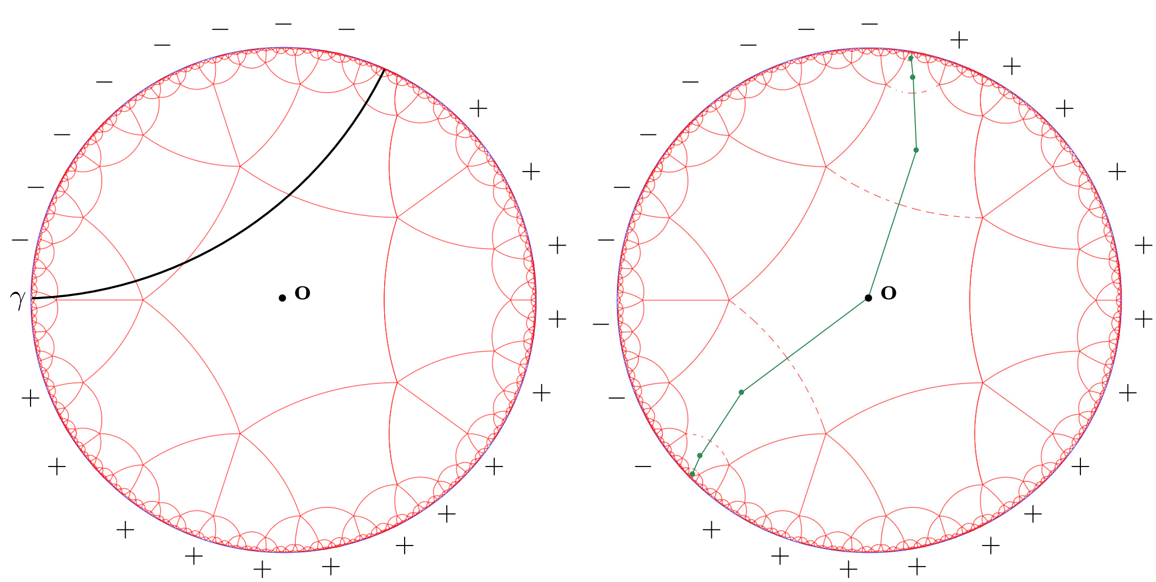

Figure 1.1: A Series–Sinai state on $\mathcal{L}_{5,5}$ selected by γ, a bi-infinite geodesic of $\mathbb{H}_2$ (in black). Right: broken edges of $\mathcal{L}_{5,5}$ (in dashed) traversed by a bi-infinite path in the dual graph (green), giving rise to an extremal state at low temperature.

2 Results

Let $\mathcal{L}_{p,q}$ be the tiling of the hyperbolic plane $\mathbb{H}_2$ such that each face is a regular p-gon and each vertex has degree q.

Our main result is a sufficient condition for well-definiteness and extremality of the low temperature Gibbs measures of the Ising model on $\mathcal{L}_{p,q}$ obtained via weak limits. These weak limits are obtained by imposing boundary conditions which have a sufficiently sparse set of frustrated bonds, but a priori no symmetry of the underlining $\mathcal{L}_{p,q}$ tilings, whence inhomogeneous. The key point is an extension of the excess energy Lemma of [9] to connected, transitive and locally finite non-amenable graphs of bounded degree, which include in particular the graph given by the 1-skeleton of the hyperbolic lattices $\mathcal{L}_{p,q}$, which we shall refer henceforth to simply by $\mathcal{L}_{p,q}$ by a slight abuse of notation. Informally, this sufficient condition measures the excess energy with respect to a reference configuration in terms of the isoperimetric constant of the graph.

For $H \subset G$ a finite subgraph of G, denoting by ∂H the usual external boundary of H, the ***isoperimetric constant*** of G is defined by

$$\mathrm{IC}_G \stackrel{\text{def}}{=} \inf\left\{\frac{|\partial\gamma|}{|\gamma|}\,;\ \gamma \text{ a finite and non-empty subgraph of } G\right\}.$$

Its explicit value on $\mathcal{L}_{p,q}$ has been famously determined by Häggström–Jonasson–Lyons [17, Theorem 4.1], and we will denote it by $\mathrm{IC}_{p,q}$ in this case, see below.

The Ising model on a locally finite graph $G = (V, E)$ is defined as follows : take $\Lambda \subset V$ be a finite subset, and ω a fixed element in $\{-1, +1\}^\Lambda$. Define the Hamiltonian $H : \{-1, +1\}^\Lambda \to \mathbb{R}$ as

$$H^\omega_\Lambda(\sigma) = -\sum_{\{i,j\}\in E\cap\Lambda} \sigma_i\sigma_j - \sum_{\{i,j\}\in E\cap\partial\Lambda} \sigma_i\omega_j$$

At inverse temperature $\beta > 0$, the Ising model in Λ with boundary condition ω is the probability measure μ^ω_Λ on $\Omega_\Lambda = \{-1, 1\}^\Lambda$ proportional to $\exp(-\beta H^\omega_\Lambda(\sigma))$. The set of infinite volume Gibbs measures of the Ising model at inverse temperature β is the set of probability measures μ on $\Omega_\Lambda = \{-1, 1\}^V$ such that the DLR equations hold:

$$\mathcal{G}_\beta = \{\mu \in \mathcal{M}_1(\{-1, 1\}^V) : \mu = \int d\mu(\omega)\mu^\omega_\Lambda \text{ for any finite } \Lambda \subset V\}$$

The set $\mathcal{G}_\beta$ is a Choquet simplex, and we denote its extremal elements by $\mathrm{ex}\mathcal{G}_\beta$, see Section 3.1. Note that the set of weak limits of sequences μ^ω_Λ with $\Lambda \uparrow V$ and a fixed boundary condition ω belong to $\mathcal{G}_\beta$. Let $\delta_{\max} \in \mathbb{N}$, and

$$\Omega_{\mathrm{GS}}(\delta_{\max}) := \{\omega \in \{-1, 1\}^V : \forall i \in V, \sharp\{j \in V : j \sim i \text{ and } \omega_i \neq \omega_j\} \leq \delta_{\max}\}$$

be the set of (infinite) spin configurations such that there are at most $\delta_{\max}$ frustrated edges emanating from any vertex. We call these $\delta_{\max}$-inhomogeneous configurations, see Definition 3.2 for details in the case of $\mathcal{L}_{p,q}$. Then the following holds:

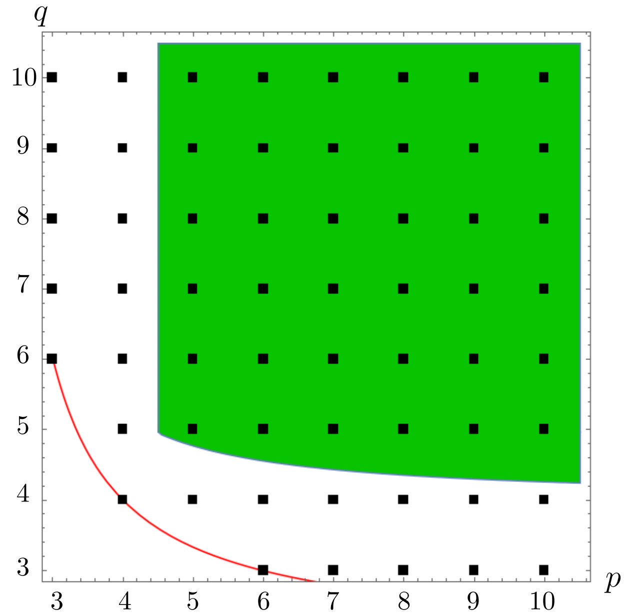

Figure 2.1: Validity region (in green) of Corollary 2.2. The red line is the curve $1/p + 1/q = 1/2$.

Theorem 2.1 (Sufficient condition for extremality)**.** *Let* IC_G *be the isoperimetric constant of the connected, transitive and locally finite graph* G*. If the following sparsity condition holds*

$$\delta_{\max} < \frac{1}{2}\,\mathrm{IC}_G \tag{2.1}$$

then, at sufficiently low temperature, for any configuration $\omega \in \Omega_{\mathrm{GS}}(\delta_{\max})$*, the infinite-volume Gibbs measure* μ^ω *of the Ising model on* G*, obtained as weak limit with boundary condition* ω*, is well-defined and extremal.*

Note that, for Theorem 2.1 to be useful, it is enough to get a (uniform and strictly positive) lower bound on IC_G, which is however a non-trivial task generally. When G is the graph given by the 1-skeleton of $\mathcal{L}_{p,q}$, the isoperimetric constant, denoted $\mathrm{IC}_{p,q}$, is explicitely known [17] and the following corollary holds:

Corollary 2.2 (Uncountably many inhomogeneous extremal states for the Ising model on $\mathcal{L}_{p,q}$)**.** *The set of extremal Gibbs measures of the Ising model on* $\mathcal{L}_{p,q}$ *is uncountable at sufficiently low temperatures for all* $p > 4$, $q \geq 3$ *such that* $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$ *and* $\mathrm{IC}_{p,q} > 2$.
This validity region is represented on Figure 2.1.

The corollary is a consequence of the observation that, in the validity region, uncountably many Dobrushin-like configurations (+ on one side of a separating line passing through the origin, − on the other side) give rise to extremal states. Indeed, interfaces of the configurations on its dual graph $\mathcal{L}_{p,q}$ can be represented as lines on the graph $\mathcal{L}_{q,p}$, see Figure 1.1. Now, following Moran [26], let us draw the "corona representation" of the graph $\mathcal{L}_{q,p}$, see Figure 2.2. The point is that for $q > 4$, $\mathcal{L}_{q,p}$ strictly includes a union of q trees of degree $(q-3)$ (which are glued at the origin). If one chooses an infinite branch in one of these trees (an end), and another one in one of the $q-3$ opposite trees (neighboring trees are forbidden), then we obtain a bi-infinite line γ which has the required property: the sparsity condition (2.1) is fulfilled for the Dobrushin configuration separated by γ.

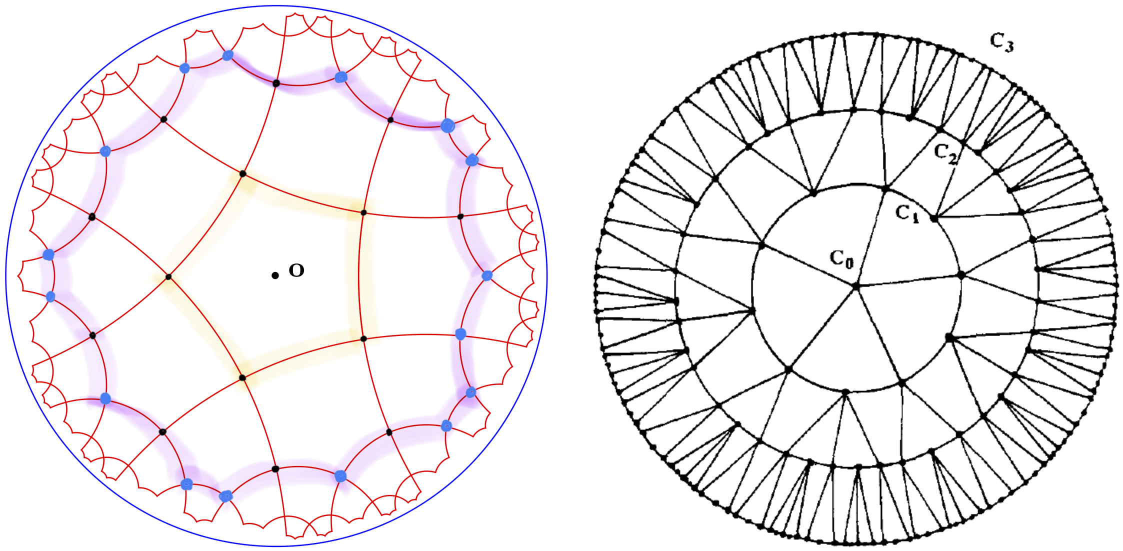

Figure 2.2: LEFT : $\mathcal{L}_{4,5}$ in red, RIGHT : Corona representation of $\mathcal{L}_{5,4}$, taken from [26].

The quantity of such good interfaces γ is uncountable by the usual Cantor set argument for the ends of trees (see *e.g.* [7, Ex. I.8 31]).

Series and Sinai left open the question of extremality of their "interface states"[1] selected by a continuum geodesics γ of $\mathbb{H}_2$, see [30, page 2, line -9]. A natural question is to wonder if all the Series–Sinai states are extremal. Theorem 2.1 above gives a sufficient condition for Series-Sinai states to be extremal: the bi-infinite geodesics γ of $\mathbb{H}_2$ should not cross more than $\frac{\mathrm{IC}_{p,q}}{2} \leq \frac{q}{2}-1$ edges emanating from any vertex of $\mathcal{L}_{p,q}$. This condition can be nicely reformulated in terms of billiard (hyperbolic) trajectories in a regular hyperbolic q-gone, by folding γ in a given fundamental domain via the reflection law. The sufficient condition is that the biliard trajectory corresponding to γ in the Dirichlet domain of $\mathcal{L}_{p,q}$ should never bounce more than $\frac{q}{2}-1$ times onto two adjacent faces. A lemma due to Mostow [27] (see also [23, Lemma 3.43]) allows us to conjecture that uncountably many Series-Sinai states are extremal. Indeed, a true geodesic of $\mathbb{H}_2$ is close to each of the Dobrushin states constructed by means of quasi-geodesics in Corollary 2.2.

Structure of the paper. Section 3.1 and Section 3.2 contain the definition of the model and some key notions. Section 3.3, which is a generalisation of the approach of [9], is devoted to the proof of Theorem 2.1. A key novelty of this paper is the generalisation of the "excess energy lemma" of [9] to rather general graphs, see Lemma 3.4, in terms of the isoperimetric constant. Section 3.4 focuses on Lobatchevsky planes $\mathcal{L}_{p,q}$, for which the isoperimetric constant is known, and allows to prove uncountability of the set of extremal Gibbs states at low temperature.

[1]We quote here a statement in [31, Page 896], which might suggest that the same authors proved the extremality of their states. However, the question is still open to the best of our knowledge.

3 Proofs

3.1 Ising model

Let $G = (V, E)$ be a simple, locally finite connected graph.

Measurable structure. We consider Ising spins: the single–spin space is $\Omega_0 = \{-1, +1\}$, endowed with the σ-algebra given by the power set $\mathcal{P}(\{-1, +1\})$ and the a priori measure $\rho_0 = \frac{1}{2}(\delta_{-1} + \delta_{+1})$.

The (infinite-volume) configuration space is $\Omega = \Omega_0^V$, with events $\mathcal{F} = \big(\mathcal{P}(\{-1, +1\})\big)^{\otimes V}$ and a priori (infinite-volume, product) measure $\rho = (\rho_0)^{\otimes V}$. For any two configurations $\sigma, \eta \in \Omega$, we consider the *partial order* $\sigma \leq \eta \iff \sigma_v \leq \eta_v$ for all $v \in V$. This partial order naturally induces a notion of increasing function: a real-valued function $f : \Omega \to \mathbb{R}$ is increasing if $\sigma \leq \eta$ implies $f(\sigma) \leq f(\eta)$ (as real numbers); and a notion of stochastic order between measures: $\mu \leq_{\rm st} \nu$ if and only if, for all increasing function f, $\mathbb{E}_\mu[f] \leq \mathbb{E}_\nu[f]$.

We will denote by $\mathcal{M}_1^+$ the set of probability measures on $(\Omega, \mathcal{F})$, which represents the macroscopic states of the system.

Microscopic finite-volume Hamiltonian. For a finite subset of vertices $\Lambda \Subset V$, a configuration $\sigma \in \Omega$ and a boundary condition ω, we consider the following *ferromagnetic*, *n.n.* Ising Hamiltonian

$$H_\Lambda^G(\sigma \mid \omega) \stackrel{\text{def}}{=} \sum_{\substack{v \sim w \\ v, w \in \Lambda}} \mathbf{1}_{\sigma_v \neq \sigma_w} + \sum_{\substack{v \sim u \\ v \in \Lambda,\, u \in \Lambda^c}} \mathbf{1}_{\sigma_v \neq \omega_u} .$$

We will denote by H the Hamiltonian with free boundary conditions, *i.e.* restricted to finite volume Hamiltonians with no interactions outside Λ.

Gibbs specification. For $\beta > 0$ (inverse temperature), we consider the following family of probability kernels $(\gamma_\Lambda)_{\Lambda \Subset V}$ defined, for all event $A \in \mathcal{F}$ and boundary condition $\omega \in \Omega$, by

$$\gamma_\Lambda(A \mid \omega) \stackrel{\text{def}}{=} \frac{1}{Z_\Lambda(\omega)} \int_\Omega \mathbf{1}_A(\sigma_\Lambda \omega_{\Lambda^c}) e^{-\beta H_\Lambda^{(p,q)}(\sigma|\omega)} \left(\rho_\Lambda \otimes \delta_{\omega_{\Lambda^c}}\right)(d\sigma)$$

where $Z_\Lambda(\omega)$ is a normalization constant, $\mathbf{1}_A(\cdot)$ is the indicator function of the event $A \in \mathcal{F}$, σ_Λ is the restriction/projection of $\sigma \in \Omega$ to the finite set Λ (analogously for ω_{Λ^c}), $\rho_\Lambda = \otimes_{v \in \Lambda} \rho_0$ is the product a priori measure over Λ and $\delta_{\omega_{\Lambda^c}}$ is the measure freezing all spins in Λ^c to the value prescribed by the boundary condition ω. Note that in a similar way, Higuchi–Yosuda built the local specification for the Ising model on the Sierpinski lattice [20].

3.2 Some definitions

Let $G = (V, E)$ be a locally finite transitive graph.

Definition 3.1 (Contour wrt σ^0 and compatibility). *Let $\sigma^0 \in \Omega$. A contour is a connected subgraph $\gamma \subset G$ s.t. $\sigma_\gamma = 1 - \sigma^0_\gamma$. Two contours γ, γ' are compatible if and only if $\gamma \triangle \gamma' = \gamma \cup \gamma'$, where $\triangle$ denotes the symmetric difference. We shall denote this compatibility relation by $\gamma \sim \gamma'$.*

For two configurations $\sigma, \eta \in \Omega$, we shall denote by $V(\sigma \triangle \eta) \subset V$ the set of sites at which σ and η disagree, which might be finite or infinite. Thus $\sigma \triangle \eta$ is a union of contours.

Then we can provide the following definition:

Definition 3.2 ($\delta_{\max}$-INHOMOGENEOUS CONFIGURATIONS)**.** *For a configuration $\sigma \in \Omega$, let $G_b(\sigma)$ be the subgraph of G induced by the set of* broken bonds *in σ, i.e. $E_b(\sigma) = \{e = (v,w) \in E\,;\ \sigma_v \cdot \sigma_w = -1\}$ (with the obvious vertex set $V_b(\sigma)$). Let $\delta_{\text{broken}}(\sigma) = \max_{v \in V_b(\sigma)} \deg(v) \leq q$. Let $0 \leq \delta_{\max} \leq q$. Then*

$$\Omega_{\text{GS}}^{p,q}(\delta_{\max}) = \{\sigma^0 \in \Omega\,;\ \delta_{\text{broken}}(\sigma^0) \leq \delta_{\max}\}) \subset \Omega$$

is called the set of $\delta_{\max}$-inhomogeneous configurations.

We shall recall from [9, Definition 3], the following:

Definition 3.3 (c-STABILITY)**.** *A configuration $\sigma^0 \in \Omega$ is c-stable if there exists a strictly positive constant $c > 0$ such that the following inequality holds*

$$H(\sigma) - H(\sigma^0) \geq c \cdot |V(\sigma \triangle \sigma^0)|$$

for all $\sigma \in \Omega$ satisfying $V(\sigma \triangle \sigma^0) \Subset V$.

3.3 Proof of Theorem 2.1

Inspired by the work [9] about finite spin models on regular trees, we shall now prove the following:

Lemma 3.4 (EXCESS ENERGY LEMMA)**.** *Let G be some connected graph. Let γ be a finite contour wrt to a given configuration $\sigma^0 \in \Omega_{\text{GS}}(\delta_{\max})$ and let $\sigma = \sigma_\gamma \sigma^0_{\gamma^c}$. Then the following holds*

$$H(\sigma) - H(\sigma^0) \geq (\text{IC}_G - 2\delta_{\max})\,|\gamma|\ ,$$

where IC_G is the isoperimetric constant of the graph and $\delta_{\max}$ is the maximal degree in the set of broken bonds.

Proof. First, by definition,

$$H(\sigma) - H(\sigma^0) = \sum_{\substack{v \in \gamma \\ w \in \gamma^c \\ v \sim w}} \left(\mathbf{1}_{\sigma_v \neq \sigma_w} - \mathbf{1}_{\sigma^0_v \neq \sigma^0_w} \right)\ .$$

Since the two configurations σ and σ^0 differ precisely at the contour γ and by definition $v \in \gamma$, $w \in \gamma^c$, we have $\sigma_v = \sigma_w$ if and only if $\sigma^0_v \neq \sigma^0_w$. Equivalently, $\mathbf{1}_{\sigma_v \neq \sigma_w} = 1 - \mathbf{1}_{\sigma^0_v \neq \sigma^0_w}$. Thus we can write

$$\sum_{\substack{v \in \gamma \\ w \in \gamma^c \\ v \sim w}} \left(\mathbf{1}_{\sigma_v \neq \sigma_w} - \mathbf{1}_{\sigma^0_v \neq \sigma^0_w} \right) = \sum_{\substack{v \in \gamma \\ w \in \gamma^c \\ v \sim w}} \left(1 - 2 \cdot \mathbf{1}_{\sigma^0_v \neq \sigma^0_w} \right) = |\partial\gamma| - 2 \sum_{\substack{v \in \gamma \\ w \in \gamma^c \\ v \sim w}} \mathbf{1}_{\sigma^0_v \neq \sigma^0_w}\ .$$

By definition of the isoperimetric constant we have a lower bound on the first term as $|\partial\gamma| \geq \text{IC}_G|\gamma|$ and an upper bound on the second term by $2\delta_{\max}|\gamma|$ (as in [9, Proof of Lemma 2]). This concludes the proof. ☐

Remark 3.5. In the case where $G = \mathcal{L}_{p,q}$, we have $\text{IC}_{p,q} = (q-2)\sqrt{1 - \frac{4}{(p-2)(q-2)}}$, see [17, Theorem 4.1]. Thus for any fixed $q \geq 3$, we have $\lim_{p \to \infty} \text{IC}_{p,q} = (q-2)$ which recovers [9, Lemma 2] (to match the notations one should put $d = q-1$ and $u = U = 1$ in [9, Eq. 3.9]). Also, $\text{IC}_{p,q} = c_{p,q} \cdot \sqrt{\frac{1}{2} - \frac{1}{p} - \frac{1}{q}}$ (for some strictly positive constant $c_{p,q}$), which equals zero only for the three Euclidean lattices.

The proof of Theorem 2.1, i.e. well-definition and extremality of μ^ω for $\omega \in \Omega_{GS}(\delta_{max})$ such that $\delta_{max} < \mathrm{IC}_G/2$ follows now from Lemma 3.4 via convergence of the cluster expansion and asymptotic decorrelation of polymer type events when β is large enough, as in [9]. For this we need the graph G to be connected, transitive (to perform the weak limit), non-amenable ($\mathrm{IC}_G > 0$), and locally finite ($\delta_{\max} < \infty$). However, to actually have a non-empty sufficient condition for extremality, one needs to at least have a positive lower bound on the Cheeger constant, and this task can be highly non-trivial. Note that in [9] the argument also extends to a wide range of SOS-like models on trees, thanks to the fact that contours have no interior on trees. Here we stick to the Ising model in order to be able to perform the usual "spin-flip trick" when proving Peierls bounds.

3.4 Proof of Corollary 2.2

3.4.1 Lobatchevsky planes

The lattice structure is provided by a regular hyperbolic tiling of the hyperbolic plane $(\mathbb{H}_2, \mathrm{d}_{\mathbb{H}_2})$ in which tiles are congruent, regular geodesic polygons with p sides of hyperbolic length equal to 1, meeting at vertices with degree q. Each such tiling is indexed by the Schläfli symbol $\{p, q\}$, for integers $p, q \geq 3$ s.t. $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$ (see [16]). With a slight abuse of notation we will denote the family of infinite q-regular metric graphs associated to the $\{p, q\}$ tilings by $\mathcal{L}_{p,q} = (V_{p,q}, E_{p,q}, F_{p,q})$.
Elements of the vertex set $V_{p,q}$ are denoted by $v, w, \ldots$. Two vertices v, w are said to be nearest neighbors, denoted $v \sim w$ (or abbreviated *n.n.*) if and only if v and w are incident to the same edge $e \in E_{p,q}$. We denote by d_{gr} the graph distance on $\mathcal{L}_{p,q}$. Equivalently, due to our embedding convention, v and w are *n.n.* if and only if $\mathrm{d}_{\mathbb{H}_2}(v, w) = 1$.

Let G be a subgraph of $(V_{p,q}, E_{p,q})$. We write $|G|$ for the number of vertices in G ($\equiv |V(G)|$). The set of *emanating edges* from G, which is a subset of $E_{p,q}$, is defined by $\partial G = \{e = (v, w) \in E_{p,q}\,;\; v \in V(G) \text{ and } w \notin V(G)\}$. The *inner boundary* of G, which is a subset of $V_{p,q}$, is defined by $\partial_{in} G = \{v \in V(G)\,;\; (v, w) \in E_{p,q} \text{ and } w \notin V(G)\}$. For G_1, G_2 two given subgraphs of $\mathcal{L}_{p,q}$, their distance is defined by $\mathrm{d}_{gr}(G_1, G_2) = \min_{\substack{v \in G_1 \\ w \in G_2}} \mathrm{d}_{gr}(v, w)$.

For $r \geq 0$ and a vertex v, the combinatorial ball of radius r centered at v is defined as follows $B(r, v) \stackrel{\text{def}}{=} \{w \in V_{p,q}\,;\; \mathrm{d}_{gr}(v, w) \leq r\}$. Analogously, the combinatorial sphere of radius r centered at v is defined by $\partial B(r, v) \stackrel{\text{def}}{=} \{w \in V_{p,q}\,;\; \mathrm{d}_{gr}(v, w) = r\}$.

3.4.2 Corona representation

$\mathcal{L}_{p,q}$ admits a *layer decomposition* (see [25, Appendix A]) which allows to get exact formulas for the number of vertices, faces and edges at or within a given layer from any given point $z \in \mathbb{H}_2$ via a transfer matrix method. This method has been discovered independently by Rietman–Nienhuis–Oitmaa [28, Eq. 2.14] and Moran [26, Pag. 159], who has also generalized it to other homogeneous tilings of $\mathbb{H}_2$. It has been extensively used by the first author together with Vanessa Jacquier and Wioletta M. Ruszel to exhibit the set of finite shape with minimal perimeter, whose exhaustion of $\mathcal{L}_{p,q}$ actually realize $\mathrm{IC}_{p,q}$, see [10]. One consequence of this layer decomposition is an exact formula for the number of vertices at or within a given layer. This number grows exponentially in the layer number with a growth rate given by the largest eigenvalue of the transfer matrix.

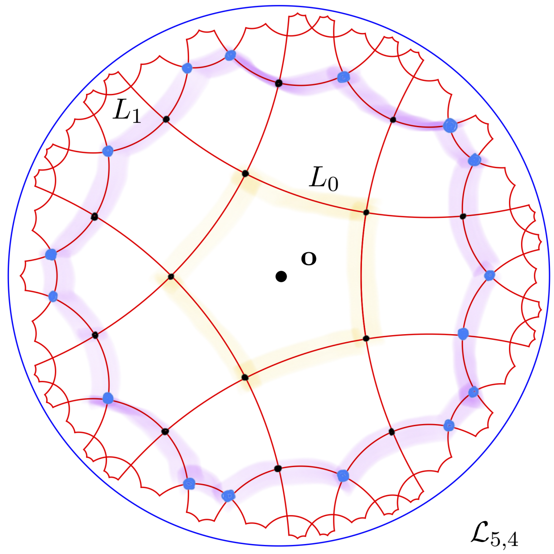

Figure 3.1: Layer decomposition for $\mathcal{L}_{5,4}$ (first two layers). In each layer, vertices are of two kind (in blue and black here) allowing counting by induction, see [28, 26] for details.

We will recall here these results for face–centered lattices $\mathcal{L}_{p,q}$ in which z is the barycenter of the fundamental polygon and it is put at the origin $\mathbf{o}$ of $\mathbb{H}_2$ (see Figure 1.1). Also, we will recall the layer decomposition for vertices of $\mathcal{L}_{p,q}$; the layer decomposition for faces and edges of $\mathcal{L}_{p,q}$ is analogous but not needed for our current purposes.

The zero layer L_0 is the set of vertices $v \in V_{p,q}$ of the face containing $\mathbf{o}$. The first layer L_1 is the set of vertices of the faces sharing a vertex or edge with the central face in L_0, which are not in L_0; and so on *ad libitum* (see Figure 3.2). Let $S_{n;p,q} = \#\{v \in V_{p,q}\,;\; v \in L_n\}$ and $B_{n;p,q} = \sum_{i=0}^{n} S_{i;p,q}$ be respectively the cardinal of layer n and the total number of vertices within layer n (layer included).

The corona representation of $\mathcal{L}_{p,q}$ goes as follows, see Figure 1.1 on the right for an example on $\mathcal{L}_{6,5}$:

- From the origin, draw p outgoing edges, and their p end-vertices in black (first generation).
- Draw a circle around them (first circle, consisting of p vertices and $p-1$ edges).
- Add $q-3$ blue vertices on each edge of the circle (triangle$\rightarrow$ q-gone).
- Draw $p-3$ outgoing edges from the black vertices of the circle, and $p-2$ outgoing edges from the blue vertices of the circle, and draw all their end-vertices in black (second generation).
- Draw a new circle around this second generation of vertices (second circle).
- Add $q-3$ blue vertices on each edge whose endpoints are attached to the same vertex of the first circle (triangle$\rightarrow$ q-gone), and $q-4$ blue vertices on each edge whose endpoints are attached to different vertices (4-gone$\rightarrow$ q-gone) of the first circle.
- Draw $p-3$ outgoing edges from the black vertices of the second circle, $p-2$ outgoing edges from the blue vertices of the second circle, and all their end-vertices in black (third generation).

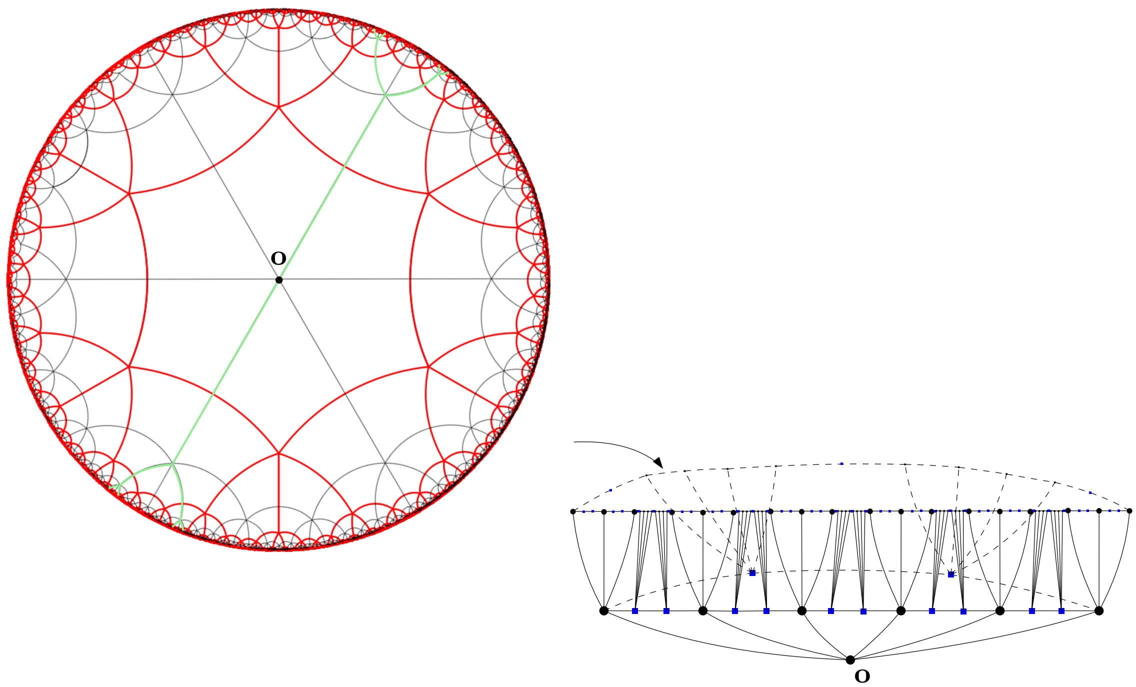

Figure 3.2: LEFT: $\mathcal{L}_{5,6}$ in red, its dual $\mathcal{L}_{6,5}$ in black, and two of the embedded ternary trees (glued at $\mathbf{o}$) in green. RIGHT : The corona representation of $\mathcal{L}_{6,5}$.

- Repeat the procedure *ad libitum*.

Note that for $p > 4$, the corona representation draws a union of p disjoint trees of degree $p-3$, which are glued at the origin, and follow the edges linking one circle to the next one.

3.4.3 Building interfaces ensuring the sparsity condition

Interfaces of the configurations on the graph $\mathcal{L}_{p,q}$ can be represented as lines on the graph $\mathcal{L}_{q,p}$, see Figure 1.1. We will now build uncountably many Dobrushin configurations which fulfill the sparsity condition (2.1). In the corona representation $\mathcal{L}_{q,p}$, we just proved that for $q > 4$, there is a union of q trees of degree $q-3$ (which are glued at the origin).

Choose an infinite branch in one of these trees, and another one in one of the $q-3$ opposite trees (neighboring trees are forbidden), then we obtain a bi-infinite line γ which do not cross more than one edge emanating from each vertex of the primal graph $\mathcal{L}_{p,q}$.

Put $+$ spins on one side of γ and $-$ spins on the other side. This forms a Dobrushin configuration $\omega^{\pm}_{\gamma}$, which belongs to $\Omega^{p,q}_{GS}(1)$.

Thus, whenever $\mathrm{IC}_{p,q} > 2$, the sparsity condition (2.1) holds, and by Theorem 2.1 the infinite-volume Gibbs measure, obtained as weak limit with boundary condition $\omega^{\pm}_{\gamma}$, is well-defined and extremal.

The quantity of such good interfaces γ is uncountable by the usual Cantor set argument (see *e.g.* [7, Example 8.11.5]). This finishes the proof of Corollary 2.2. $\square$

3.5 Perspectives

We leave the generalization of this question to the ferromagnetic nearest neighbors Ising model defined on Cayley graphs of finitely generated co-compact group of isometries of $\mathbb{H}_d$ ($d \geq 2$) to future work. We also leave the study of the multivariate generating functions associated to the layer construction of Rietman–Nienhuis–Oitmaa/Moran to future work.

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