RAPPORT DE STAGE

Comparison of different Euclidean Random Assignment Problems

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Résumé

Dans ce rapport, nous avons étudié le problème d'assignation lorsque deux ensembles de points ayant le même nombre sont distribués aléatoirement. L'objectif est de trouver l'assignation optimal, c'est-à-dire l'assignation avec le coût total minimum, qui est basé sur les hypothèses spécifiques du problème, et nous souhaitons étudier ses propriétés statistiques. Les hypothèses du problème comprennent la définition de la fonction de coût d'assignation, les domaines sur lesquels les deux ensembles de points sont distribués, la loi de distribution des points et le nombre de points. Ce problème de mécanique statistique, appelé "problème d'assignation aléatoire Euclidien" [1], est lié aux travaux de Mézard et Parisi dans les années 1980 sur les modèles mathématiques des verres de spin [2, 3].

Dans la première section, nous avons introduit plusieurs problèmes spécifiques d'intérêt, qui se distinguent de ceux déjà étudiés par l'une des hypothèses, c'est-à-dire les domaines, notés (Ω_b, Ω_r). Les domaines que nous avons considérés sont des domaines glués unidimensionnels, par exemple les graphes en étoile, et certains domaines convexes bidimensionnels, par exemple les ellipses et les triangles. Nous avons ensuite résumé notre étude de ces problèmes en étudiant l'effet du changement du domaine sur l'assignation optimale. Plus précisément, en se basant sur le cas $\Omega_b = \Omega_r$, nous étudions deux types de changements, tout d'abord, le changement du domaine de distribution d'un seul ensemble de points, c'est-à-dire en comparant un problème sur (Ω, Ω) et un sur (Ω, Ω'), et deuxièmement, en changeant les deux domaines de la même manière, c'est-à-dire en comparant un problème sur (Ω, Ω), et un sur (Ω', Ω').

Dans Section 2, nous avons passé en revue certains résultats existants sur les problèmes de dimension basse qui sont utiles pour notre étude. Dans Section 3, nous avons fourni un théorème permettant de comparer le coût total minimum de deux problèmes d'assignation aléatoire euclidiens différents avec le même nombre de points, en utilisant l'ordre stochastique des coûts d'une des paires dans ces deux problèmes. Ce théorème peut être utilisé pour comparer un problème sur (Ω_1, Ω_2) et un autre sur (Ω_3, Ω_4), qui comprend les deux cas mentionnés précédemment.

La section suivante, basée sur les problèmes introduits dans Section 1, présente deux applications du théorème, y compris des études du problème sur le graphe en k-étoile unidimensionnel et des comparaisons entre certains problèmes en dimensions supérieures. Dans la dernière section, nous avons discuté des limitations et d'autres applications possibles du théorème, et présenté quelques conjectures.

Mots-clés

Problème d'assignation aléatoire euclidienne, Ordre stochastique, Graphe en étoile, Domaine convexe bidimensionnel, Simulation numérique en Python

Abstract

In this report, we studied the one-to-one matching problem when two sets of points with the same number are randomly distributed. The aim was to find the optimal match, i.e. the match with the minimum total cost, which is based on the specific assumptions of the problem, and we wanted to investigate its statistical properties. The assumptions of the problem include the definition of a cost function, the domains (denoted as Ω_b and Ω_r) over which the two sets of points are distributed, the distribution law of the points, and the number of points. This statistical mechanics problem, called "Euclidean Random Assignment Problem" [1], is related to the work of Mézard and Parisi in the 1980s on mathematical models of spin glasses [2, 3].

In the first section, we introduced several specific problems of interest, which are distinguished from those already studied by one of the assumptions, i.e. Ω_b and Ω_r. The domains we considered are one-dimensional glued domains, e.g. star graph, and some two-dimensional convex domains, e.g. ellipses and triangles. We then summarized our study of these problems as studying the effect of changing domains on the optimal match. Specifically, based on the $\Omega_b = \Omega_r$ case, we study two kinds of changes, firstly, changing the distribution range of only one point set, i.e. comparing a problem on (Ω, Ω) and one on (Ω, Ω'), and secondly, changing both domains in the same way, i.e. comparing a problem on (Ω, Ω) and one on (Ω', Ω').

We reviewed in Section 2 some existing findings on low-dimensional problems which are useful for our study. In Section 3, we provided a theorem to compare the minimum total cost of two different Euclidean Random Assignment Problems with the same number of points, using the stochastic order of the costs of one of the pairs in these two problems. This theorem can be used to compare a problem on (Ω_1, Ω_2) and one on (Ω_3, Ω_4), which includes the two cases mentioned earlier.

The subsequent section, based on the problems introduced in Section 1, provide two applications of the theorem, including studies of the problem on the one-dimensional k-star graph and comparisons between some problems in higher dimensions. In the final section of this report, we discussed the limitations and more possible applications of the theorem, and presented some conjectures.

Keywords

Euclidean Random Assignment Problem, Stochastic order, Star graph, Two-dimensional convex domain, Numerical simulation in Python

Contents

1 Introduction

Suppose that there are 5 mines mining iron ore in city A and 5 factories in city B that use the iron ore produced by the mines. The transport cost from a mine to a factory is determined by the distance between these two sites. In addition, each mine can only supply one factory. We want to find the optimal transport plan that minimises the total transport cost of the 5 transport routes [4].

We notice that the solutions in this problem are countable (and the number of them is equal to 120), that is, if all the possibilities are shown, the optimal solution can be found. There are many other problems having this characteristic, and they are known generally as combinatorial optimization problems [5]. Some of them, like traveling salesman problem (TSP) [6], job-shop scheduling problem (JSSP) [7], Bin Packing Problem (BPP) [8, chapter 18] and K-satisfiability (K-SAT) [9], have long been questions of great interest in a wide range of fields.

The problem introduced at the beginning is an example of the assignment problem [10], a classical type of combinatorial optimisation problem. The assignment problem has received increased attention across a number of disciplines in recent years, like in education [11], transportation [12], and healthcare [13]. Our work focuses on a type of this problem in which objects are random points: the Euclidean Random Assignment Problem (ERAP) [1].

1.1 Euclidean Random Assignment Problem

Consider an n-sample $\mathcal{B} = (B_1, \dots, B_n)$ of i.i.d. random variables of law μ_B over the defined domain Ω_b, and another n-sample $\mathcal{R} = (R_1, \dots, R_n)$ of i.i.d. r.v.s of law μ_R over Ω_r. We are interested in the statistical properties, the random variable

$$\mathcal{H}_{\text{opt}}^{\Omega_b,\Omega_r} := \min_{\pi \in S_n} \sum_{i=1}^{n} D^p(B_i, R_{\pi(i)}) \,, \tag{1}$$

where S_n is the symmetric group over n elements, and $D(B_1, R_1)$ is the Euclidean distance between points B_1 and R_1. In some cases, we use the notation $\mathcal{H}_{\text{opt}}^{\Omega_b,\Omega_r,n}$ to stress the dependence on n. However, without causing any misunderstanding, we will also abbreviate it to $\mathcal{H}_{\text{opt}}$.

For a specific case of ERAP, we specify some assumptions. First, the domain Ω_b and Ω_r in which the points are distributed. The most easily studied scenarios are those on one dimension. We will review some research results in the different dimensions in Section 2. The widely studied supposition is that $\Omega_b = \Omega_r$, but in this work we also discuss the case of $\Omega_b \neq \Omega_r$. Second, the law μ_B, μ_R for random variables. Typically, the random variables follow the continuous uniform distribution, but other situations have also attracted attention, such as where a point set $\mathcal{B}$ is a deterministic grid on the domain [14]. Third, the exponent p. Depending on the value of p, and especially on the three groups: $p > 1$, $0 < p < 1$, $p < 0$, the optimal assignment will show different properties. The previous findings in different cases will also be recalled in Section 2. Last, the number of points (n). In addition to studies relating to specific n, the value $\mathbb{E}[\mathcal{H}_{\text{opt}}]$ as n tends to infinity is also worth being studied, as it can help us understand the relationship between the ERAP and its continuous counterpart, the Monge-Kantorovich [15].

ERAP can be used in statistical physics as a toy model for the study of "spin-glass" [2, 3]. The individual atomic bonds in a spin glass consist of almost equal numbers of two types of bonds (ferromagnetic bonds and antiferromagnetic bonds), which can be represented by $\mathcal{B}$ and $\mathcal{R}$. The main focus is on the typical properties of systems made up of these particles in the limit of low temperature, i.e. the lowest energy state, as denoted by $\mathcal{H}_{\text{opt}}$.

1.2 Contributions of the work

Some of our work developed into a short paper [16], in which we prove Theorem 1 to compare $\mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega_1,\Omega_2}]$ and $\mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega_3,\Omega_4}]$ by using stochastic order of $c(B_1^{(\Omega_1)}, R_1^{(\Omega_2)})$ and $c(B_1^{(\Omega_3)}, R_1^{(\Omega_4)})$, i.e., the cost functions of a single couple (B_1, R_1) without the assignment constraint. As mentioned in the former part, this theorem also applies to the case of $\Omega_b \neq \Omega_r$. In [16], we then used the theorem to discuss some special structures. Our interest in these structures will be introduced at the end of this section. More detailed proofs and additional comments will be given in this report. All python code used to visualise specific problems in this report is publicly available[1].

1.3 Motivation and structure of the report

It is following numerical experiments on ERAPs that has driven this research. Firstly, for two-dimensional problems, the impact of the shape of the domains on the total cost has attracted attention. An example is shown in Figure 1.1-1.3, that is, if other assumptions are the same and $\Omega_b = \Omega_r := \Omega$ are disc, square, and triangle respectively, we might be able to compare the $\mathbb{E}[\mathcal{H}_{\text{opt}}]$ corresponding to each figure.

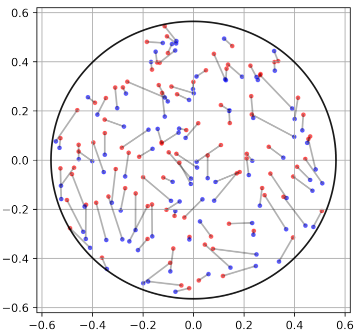

Figure 1.1: Ω: disc, $n = 100$

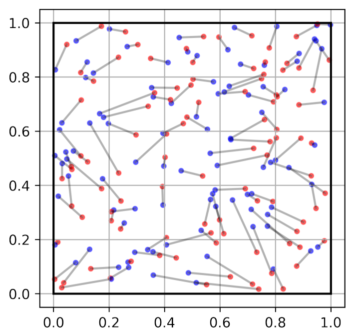

Figure 1.2: Ω: square, $n = 100$

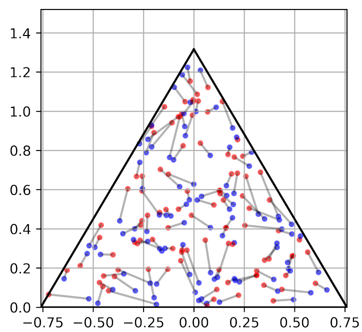

Figure 1.3: Ω: triangle, $n = 100$

Similarly, another example is, when we consider that Ω are ellipses of the same area but different shapes, whether the value of $\mathbb{E}[\mathcal{H}_{\text{opt}}]$ depends on the shape (see Figure 1.4).

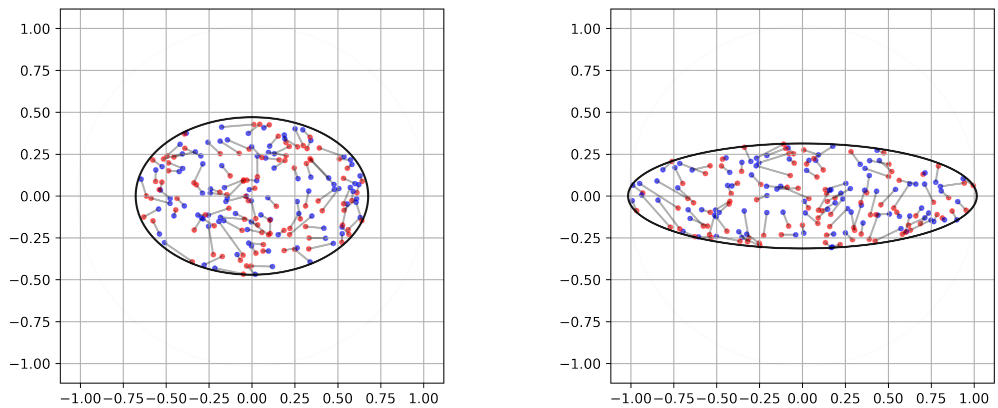

Figure 1.4: Ω: ellipses of the same area but different shapes, $n = 100$

In addition, Numerical experiments on the ERAP with glued domains have also driven our work. In this report, we mainly discuss one-dimensional problem with glued domains. Two-dimensional problem with glued domains are defined similarly, but the applications are more challenging, and we will also discuss them shortly in Section 5.

For one-dimensional problems, a great deal of previous research into ERAP has focused on the case where Ω_b and Ω_r are line segments (see Figure 1.5(a)). Based on this simple structure, the glued domains are star graphs, as exemplified by Figure 1.5(b-d). There are practical application scenarios for such structure. For example for the problem at the beginning of this report, it corresponds to the case where the factories and the mines are located on different roads with (one or more) crossings. This leads naturally to the definition of distance, where transport between two points on different sides needs to pass through the crossing points. We will present detailed definition in Section 4.1.3.

We note that for the star graph defined here, the number of sides and the number of vertices can be arbitrary positive integers and the length of each side can be different. Similar to this structure, we

[1] https://github.com/matteodachille/ERAPs2d/tree/Visualisation-examples-from-Yuqi's-M2-report

can define wormholes on line segment. As shown in Figure 1.6(a), the transport distance between the red point and the blue point is the sum of the lengths of the two grey line segments ($l_1 + l_2$). In other words, wormholes t_1 and t_2 can reduce the transport distance between points. An equivalent structure to Figure 1.6(a) is Figure 1.6(b), which contains a closure of the ring rather than two wormholes. That is, the distance between the same red point and blue point can be represented by l. We mention this because there have been studies of ERAP on a circle [17] (see Figure 1.6(d)), and particularly, this equates to the wormholes being at exactly the two end points of the line segment (see Figure 1.6(c)).

There are two ideas that can help us examine these problems mentioned earlier. Firstly, contrast the problem on a star graphs (e.g. Figure 1.5(b)) with the problem on a line segment (e.g. Figure 1.5(a)). Secondly, $\mathcal{H}_{\text{opt}}$ on the star graph is mainly related to the transport costs on each side. A detailed explanation can be found in [18]. However, due to the fact that the number of red points on each branch may not equal the number of blue points, this sub-problem differs from the one shown in Figure 1.5(a): for the former, $\Omega_b = \Omega_r$; for the latter, $\Omega_b \neq \Omega_r$. We will discuss this in Section 4.1.2.

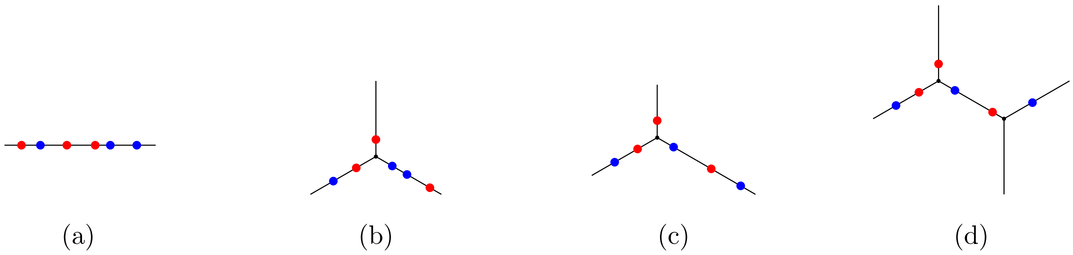

Figure 1.5: Star graphs

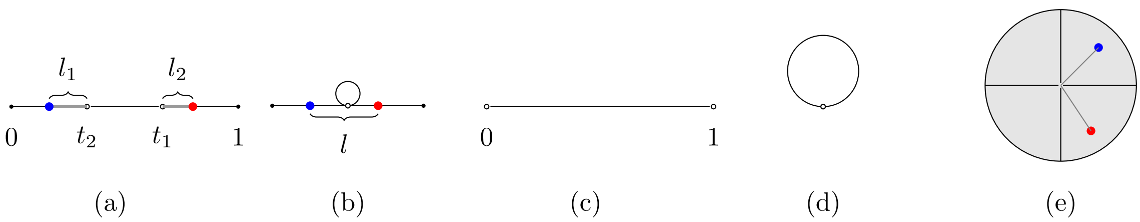

Figure 1.6: Glued domains

Let us now turn to the two-dimensional problem with glued domains. We can also call this problem ERAP on 2-dimensional domain with barriers.

Let $\mathcal{B}$ and $\mathcal{R}$ be i.i.d. random variables over $\bigcup_{1\leqslant j\leqslant k} \Omega_i$, and $\Omega_1, \Omega_2, ...\Omega_k$ be k convex sets in two-dimensional Euclidean space, identical in shape, of equal areas $A(\Omega_j) = \frac{1}{k}$, $j = 1, \dots, k$, and sharing exactly one point w which for the sake of definitiveness we take to be $(0, 0)$. The distance between two points, $a = (x_a, y_a)$ and $b = (x_b, y_b)$, in $\bigcup_{1\leqslant j\leqslant k} \Omega_i$ is defined by

$$D(a,b) = \begin{cases} \sqrt{(x_a - x_b)^2 + (y_a - y_b)^2}, & \text{if } a, b \in \Omega_j, j = 1, \dots, k \\ \sqrt{x_a^2 + y_a^2} + \sqrt{x_b^2 + y_b^2}, & \text{otherwise.} \end{cases}$$

Figure 1.6(e) shows an example where the grey line segments indicate the transport path with the shortest distance.

For the two ideas presented earlier for studying one-dimensional problems with glued domains, the first one, i.e. directly comparing the problem on a glued domain with the problem on a convex domain, is difficult. So we discuss the second idea here.

Once the solution π^* is found, we consider a part of connections appearing in the optimal matching:

$$S_1 = \{(b_i, r_{\pi^*(i)}) : \text{ the blue point } b_i \text{ and the red point } r_{\pi^*(i)} \text{ are in } \Omega_1\},$$

The assumptions of this sub-problem can be described in another way: $\mathcal{B}$ is an n-samples of i.i.d. random variables over a domain Ω, $|\Omega| = 1$, and $\mathcal{R}$ is another n-samples of i.i.d. random variables over the domain $\Omega' \subset \Omega$, $|\Omega \setminus \Omega'| = \frac{1}{\sqrt{n}}$. Specifically, we would likely consider three cases:

1. $\Omega = [0,1]^2$ and $\Omega' = [0,1] \times [0, 1 - \frac{1}{\sqrt{n}}]$. (See Figure B.1)
2. $\Omega = B(0, \frac{1}{\sqrt{\pi}})$ and $\Omega' = B(0, \frac{1}{\sqrt{\pi}}\sqrt{1 - \frac{1}{\sqrt{n}}})$. (See Figure B.2)
3. $\Omega = B(0, \frac{1}{\sqrt{\pi}})$ and $\Omega' = B(0, \frac{1}{\sqrt{\pi}}) \setminus B(0, \frac{n^{-1/4}}{\sqrt{\pi}})$. (See Figure B.3)

The third case is the one most relevant to the two-dimensional problem with glued domains, because in optimal matching, red (blue) points close to the center are more likely to be connected to blue (red) points in different divisions.

In summary, the problems we are interested in can be divided into two types. That is, setting $\Omega_b = \Omega_r = \Omega \neq \Omega'$, we study the effects of change of domain on $\mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega,\Omega}]$ in the following two cases,

1. Both domains are changed to the same one, i.e., compare $\mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega,\Omega}]$ and $\mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega',\Omega'}]$.
2. There is only one domain changed, i.e., compare $\mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega,\Omega}]$ and $\mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega,\Omega'}]$.

This report is divided into six distinct sections. The second section gives a brief review of the ERAPs, including the conclusions in the one-dimensional and two-dimensional problems and the connection between ERAPs and Monge-Kantorovich problem. The third section summarises the main finding of our work. We will prove an inequality which is valid for a more general case than the two types of problem above. Some applications of the finding are discussed in Section 4, with regard to the cases presented earlier. We have not resolved all the problems mentioned earlier in this report. In the final section, we will present some ideas as perspectives for future research.

2 Literature review

Since the work of Mézard and Parisi [2, 3] that used the random assignment problem as a model to study spin glasses, a number of studies have begun to examine the statistical properties under different assumptions [4, 14, 17, 19–32].

In this section, we review previous work in various dimensions of domain. Before that, let us review a thought which is useful when we discuss universality in the ERAP. A discrete counterpart to the Monge-Kantorovich problem is the ERAP. And we remark that

$$\lim_{n\to+\infty} \frac{1}{n}\mathbb{E}\left[\mathcal{H}_{\mathrm{opt}}^{\Omega_b,\Omega_r}\right] = W_p^p(\rho_{\mathcal{B}}, \rho_{\mathcal{R}}) \ ,$$

where $\rho_{\mathcal{B}}, \rho_{\mathcal{R}}$ are the empirical measures of $\mathcal{B}$ and $\mathcal{R}$ and $W_p^p(\mu,\nu)$ is the p-th power of the Wasserstein distance W_p among probability measures μ, ν (for the relevant definitions, see *e.g.* [33]).

2.1 1-dimensional Euclidean Random Assignment Problems

To date, several studies (e.g. [14, 17, 19]) have investigated the case of the distribution of points on a line segment or on a circle, with $p \geqslant 1$. In the case of a line segment, for a convex cost function, the optimal matching has a very clear construction, that is, ordering the red and blue points respectively according to their distance from one of the endpoints of the line segment, the i-th red point match the i-th blue point (for all $1 \leqslant i \leqslant n$). An example is illustrated in Figure 2.1 (a). This finding has allowed for further research into the minimum total cost $\mathcal{H}_{\mathrm{opt}}$. A similar conclusion holds true when the points are distributed on a circle (for example, see Figure 2.1 (b)). There are also studies on $p < 0$ or $0 < p \leqslant 1$, e.g. [4, 20], and these cases are more challenging than the problem with convex cost functions. In the rest of this report, we only discuss the case of convex cost functions. More reviews of studies corresponding to other cost functions can be found in [1]. For another item in the hypothesis, namely the distribution of points, some problems of non-uniform distribution have also been studied [21–24].

(a) (b)

Figure 2.1: Examples of one-dimensional ERAP.

2.2 2-dimensional Euclidean Random Assignment Problems

Compared to one-dimensional problems, two-dimensional problems are more extensively studied, and one reason for this is that there are more classical two-dimensional domains that have attracted attention. Benedetto *et al.* (2021) [25] studied the problem on various families of surfaces, including unit rectangle, flat torus, disc, cone, unit sphere and real projective sphere. On these surfaces, if $p = 2$, we have the following fundamental result,

$$\lim_{n\to+\infty} \mathbb{E}\left[\mathcal{H}_{\mathrm{opt}}^{\Omega,\Omega}\right] \simeq \frac{|\Omega|}{2\pi}\log n + o(\log n) \tag{2}$$

Here, $|\Omega|$ means the area of Ω and "$\simeq$" means the asymptotic equivalence. This result was presented in [34] in a more generalized form,

$$\lim_{n\to+\infty} \mathbb{E}\left[\mathcal{H}_{\mathrm{opt}}^{\Omega,\Omega}\right] \simeq c_p n^{1-\frac{p}{2}} (\log n)^{\frac{p}{2}} (1 + o(1)) \, , \quad p \geqslant 1 \, . \tag{3}$$

Then, it is only since the work of Caracciolo *et al.* (2014) [26], in which they proposed a linearisation ansatz of the Monge-Ampère equation to the Poisson equation, that research into the proof of this conclusion has gained momentum. Now, eq.(2) has been proved in [27, 28] via PDE methods to hold for any compact 2-dimensional Riemannian manifold without boundary. More recently, there has been an increasing amount of literature on PDE methods for studying this $2d$ problem [29–31].

In all the studies reviewed here for eq.(2), little is known about anything except the leading-order term. In [25], the authors have begun to examine the sub-leading-order term on each surface.

In addition to these low-dimensional cases, a number of authors have considered the case in dimension $\geqslant 3$, for example Talagrand (1994) [32].

3 Basic definitions and main result

Stochastic dominance [35, 36] is useful in a variety of probability theory fields, including stochastic processes [37, Section 9], discrete probability distributions [38], and decision theory [39]. In this section, we first review some basic definitions and properties of stochastic dominance, in particular about the stochastic order of random variables, then using them to present our main theorem.

3.1 Stochastic order of random variables

Definition 1. [First-degree and Second-degree stochastic dominance]
Two random variables X and Y are in *stochastic order*, specifically, X is said to be *first-degree stochastically dominant* over Y, written $Y \leq_{\mathrm{st}} X$, if

$$\mathbb{P}(X \geqslant t) \geqslant \mathbb{P}(Y \geqslant t), \quad \forall t \in \mathbb{R}\,. \tag{4}$$

X is said to be *second-degree stochastically dominant* over Y, i.e. $Y \leq_{\mathrm{ssd}} X$, if and only if

$$\int_{-\infty}^{u} \mathbb{P}(X \leqslant t)\mathrm{d}t \leqslant \int_{-\infty}^{u} \mathbb{P}(Y \leqslant t)\mathrm{d}t,\ \forall u \in \mathbb{R}. \tag{5}$$

Proposition 1. There are some other ways to rewrite eq.(4):

1. $F_X(t) \leqslant F_Y(t),\ \forall t \in \mathbb{R}$, where $F_X(t)$ and $F_Y(t)$ are the cumulative distribution functions;
2. $\mathbb{P}(X \in T) \geqslant \mathbb{P}(Y \in T), \forall T \subset \mathbb{R}$, where T is an open set, or T is a half-open set including only the right endpoints;
3. $\mathbb{E}[I_T(X)] \geqslant \mathbb{E}[I_T(Y)], \forall T \subset \mathbb{R}$, where I_T means the indicator function of T, and T is an open set, or T is a half-open set including only the right endpoints;
4. $\mathbb{E}[\phi(X)] \geqslant \mathbb{E}[\phi(Y)]$, for all increasing functions ϕ for which the expectations exist;
5. $\int_u^\infty \mathbb{P}(X > t)\mathrm{d}t - \int_u^\infty \mathbb{P}(Y > t)\mathrm{d}t$ is decreasing in $u \in (-\infty, \infty)$.

3.2 Main theorem

The following theorem is the main result of our work, that is, using stochastic order to compare different Euclidean Random Assignment Problems.

Theorem 1. Let $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ be four domains. $\mathcal{B}^{(\Omega_1)}$ are n i.i.d. r.v.s over Ω_1. Analogously, $\mathcal{R}^{(\Omega_2)}$, $\mathcal{B}^{(\Omega_3)}$ and $\mathcal{R}^{(\Omega_4)}$ are respectively n i.i.d. r.v.s over Ω_2, Ω_3, and Ω_4 (see Figure 3.1-3.2). Consider the cost function defined by

$$\begin{aligned} c\ &: \Omega_1 \times \Omega_2 \cup \Omega_3 \times \Omega_4 \to \mathbb{R}^+ \\ &(x, y) \qquad\quad \mapsto c(x, y), \end{aligned}$$

and let $F_{c(x,y)}(t)$ be its cumulative distribution function. Assume that, if we take arbitrary points, one in each set $\mathcal{B}^{(\Omega_1)}$, $\mathcal{R}^{(\Omega_2)}$, $\mathcal{B}^{(\Omega_3)}$ and $\mathcal{R}^{(\Omega_4)}$, and denote them separately as $B^{(\Omega_1)}$, $R^{(\Omega_2)}$, $B^{(\Omega_3)}$ and $R^{(\Omega_4)}$ (see Figure 3.3-3.4), then the following inequality holds

$$c\left(B^{(\Omega_3)}, R^{(\Omega_4)}\right) \leq_{\mathrm{st}} c\left(B^{(\Omega_1)}, R^{(\Omega_2)}\right)\,. \tag{6}$$

Then

$$\mathbb{E}[\mathcal{H}_{\mathrm{opt}}^{\Omega_3,\Omega_4,n}] \leqslant \mathbb{E}[\mathcal{H}_{\mathrm{opt}}^{\Omega_1,\Omega_2,n}],\ \forall n \geqslant 1. \tag{7}$$

Remark 1.

(i) We now present the sketch of the proof, which is detailed in the Appendix A.1. Eq.(6) leads to

$$\sum_{i=1}^{n} c(B_i^{(\Omega_3)}, R_{\pi(i)}^{(\Omega_4)}) \leq_{\mathrm{st}} \sum_{i=1}^{n} c(B_i^{(\Omega_1)}, R_{\pi'(i)}^{(\Omega_2)}), \forall \pi, \pi' \in S_n, \tag{8}$$

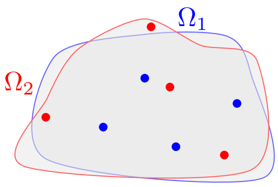

Figure 3.1: $\mathcal{B}^{(\Omega_1)} \subset \Omega_1, \mathcal{R}^{(\Omega_2)} \subset \Omega_2$

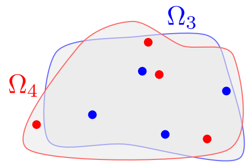

Figure 3.2: $\mathcal{B}^{(\Omega_3)} \subset \Omega_3, \mathcal{R}^{(\Omega_4)} \subset \Omega_4$

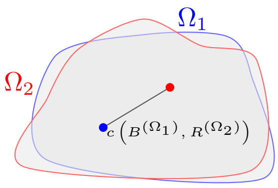

Figure 3.3: $B^{(\Omega_1)} \in \Omega_1, R^{(\Omega_2)} \in \Omega_2$

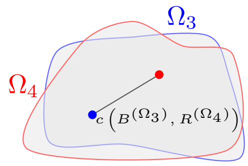

Figure 3.4: $B^{(\Omega_3)} \in \Omega_3, R^{(\Omega_4)} \in \Omega_4$

then,

$$\min\left\{\sum_{i=1}^{n} c(B_i^{(\Omega_3)}, R_{\pi(i)}^{(\Omega_4)}), \forall \pi \in S_n\right\} \leq_{\text{st}} \min\left\{\sum_{i=1}^{n} c(B_i^{(\Omega_1)}, R_{\pi(i)}^{(\Omega_2)}), \forall \pi \in S_n\right\}. \tag{9}$$

Hence,

$$\mathbb{E}\left[\min_{\pi\in S_n}\sum_{i=1}^{n} c(B_i^{(\Omega_1)}, R_{\pi(i)}^{(\Omega_2)})\right] \geqslant \mathbb{E}\left[\min_{\pi\in S_n}\sum_{i=1}^{n} c(B_i^{(\Omega_3)}, R_{\pi(i)}^{(\Omega_4)})\right], \ \forall n \geqslant 1, \tag{10}$$

So we obtain eq.(7).

(ii) The domains $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ in the hypothesis are generalised, that means, for instance, the boundaries could be less than regular or the domains could be not bounded, as eq.(6) is a very strong assumption. This theorem holds even if the cost function $c(x, y)$ can be equal to $+\infty$.

(iii) If neither $c\left(B^{(\Omega_1)}, R^{(\Omega_2)}\right)$ nor $c\left(B^{(\Omega_3)}, R^{(\Omega_4)}\right)$ stochastically dominates the other, then the direction of the inequality in eq.(7) generally depends on n. We illustrate in section 4.1.3 a one-dimensional example.

(iv) Even if $\Omega_1 = \Omega_3$ and $\Omega_2 = \Omega_4$, it is possible that $c\left(B^{(\Omega_1)}, R^{(\Omega_2)}\right)$ is not equal to $c\left(B^{(\Omega_3)}, R^{(\Omega_4)}\right)$ by definition of the cost function $c(x, y)$. For instance, one can set $c\left(B^{(\Omega_1)}, R^{(\Omega_2)}\right) = D^2\left(B^{(\Omega_3)}, R^{(\Omega_4)}\right)$ and $c\left(B^{(\Omega_3)}, R^{(\Omega_4)}\right) = D^4\left(B^{(\Omega_3)}, R^{(\Omega_4)}\right)$.

(v) Theorem 1 holds for *any* choice of distribution of points (as long as $B_i(R_i) \perp\!\!\!\perp B_j(R_j)$, $i \neq j$, $i, j = 1, \ldots, n$).

(vi) As in the case of Remark 3 (iii) as an example, if we have equality

$$F_{c\left(B^{(\Omega_3)}, R^{(\Omega_4)}\right)}(t) = F_{c\left(B^{(\Omega_1)}, R^{(\Omega_2)}\right)}(t), \ \forall t \geqslant 0,$$

then we get equality among *all moments*

$$\mathbb{E}\left[\left(\mathcal{H}_{\text{opt}}^{\Omega_1,\Omega_2,n}\right)^m\right] = \mathbb{E}\left[\left(\mathcal{H}_{\text{opt}}^{\Omega_3,\Omega_4,n}\right)^m\right], \ \forall n \geqslant 1, \ m \in \mathbb{N}.$$

Now, we illustrate this theorem using the first example mentioned in Section 1. More examples are discussed in the next section.

Example 1. From the numerical results, we can obtain that, by letting Ω be a disc, a square or an equilateral triangle of the same area, $D\left(B^{(\Omega)}, R^{(\Omega)}\right)$ are in stochastic order, in other words, the $F_{D\left(B^{(\Omega)}, R^{(\Omega)}\right)}(t)$ obtained decreases sequentially for all $t \in \mathbb{R}$ (as shown in Figure 3.5), and thus, setting

$c\left(B^{(\Omega)}, R^{(\Omega)}\right) = D^2\left(B^{(\Omega)}, R^{(\Omega)}\right)$, the $\mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega,\Omega}]$ increases sequentially for every $n \in \mathbb{N}^*$ (see Figure 3.6). This corresponds to Theorem 1. Figure 1.1-1.3 illustrate the matching results of one simulation.

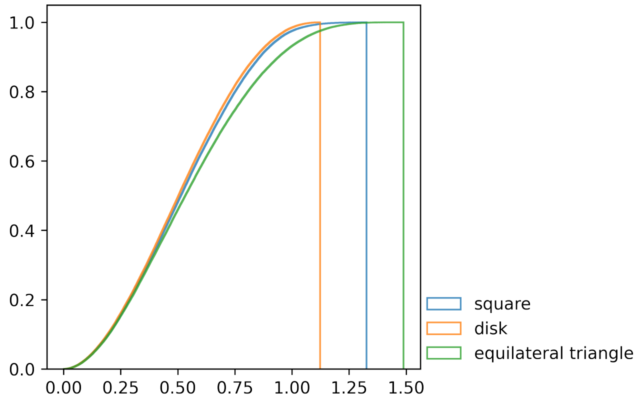

Figure 3.5: Cumulative distribution functions of $D\left(B^{(\Omega)}, R^{(\Omega)}\right)$ with Ω being a disc, a square or an equilateral triangle of area 1.

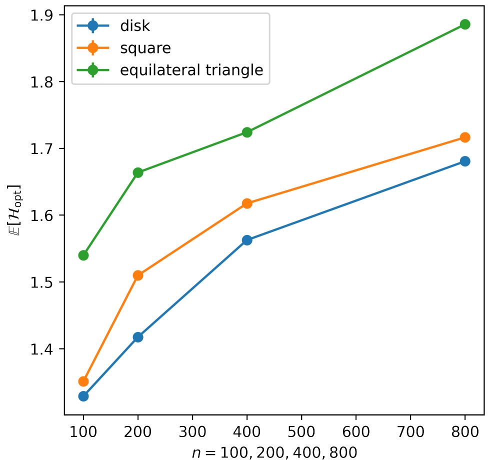

Figure 3.6: $\mathbb{E}_n[\mathcal{H}_{\text{opt}}]$ as a function of n with SEM (standard error of the mean) error bars. The errors are too small to be visible. The number of simulated instances is 1000.

4 Applications of Theorem 1

We show in this section how Theorem 1 can be used to study more one-dimensional problems and two-dimensional problems mentioned in Section 1.

4.1 Applications of Theorem 1 in dimension $d = 1$

Before the applications, our discussion starts with possibly the simplest case, that is, both domains Ω_b and Ω_r are line segments. In this part, we assume that $\mathcal{B}$ ($\mathcal{R}$) are uniformly distributed unless otherwise stated.

4.1.1 Ω_b and Ω_r are line segments: Results valid in general

In dimension $d = 1$, one can get a lot more information due to the particularly simple combinatorial properties of the optimal permutation π_{opt}. For example, As we recalled in the previous section, for an ERAP over an interval, if $p \geq 1$, (strict) convexity of the cost function and optimality imply $\pi_{\text{opt}}(i) = i$, $\forall i = 1, \ldots, n$, if the points are sorted in natural order (see *e.g.* [1, 17]). More precisely, in the case $\Omega_r = \Omega_b = \Omega = [0, 1]$, the reformulation of ERAP in terms of generalized Selberg integrals [40] allows to write, for $p \geq 1$,

$$\mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega,\Omega,n}] = \frac{\Gamma\left(1+\frac{p}{2}\right)}{(p+1)}\, n\, \frac{\Gamma(n+1)}{\Gamma\left(n+1+\frac{p}{2}\right)}, \qquad \forall n \in \mathbb{N}. \tag{11}$$

We can thus address (here in the case $p = 2$ but with minor modifications for generic $p \geq 1$, p even) the evaluation of $\mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega_b,\Omega_r}]$, which we resume in the following.

Proposition 2. Let $\Omega_b = [0, l_b]$ and $\Omega_r = [0, l_r]$, for $l_b,\ l_r > 0$, and let $\Omega = [0, 1]$. Then, at $p = 2$,

$$\mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega_b,\Omega_r}] = \left((n+1)(l_b^2 + l_r^2) - (2n+1)l_b l_r\right) \mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega,\Omega}],\ \forall n \in \mathbb{N}. \tag{12}$$

Proof of Proposition 2. Let $\rho_b,\ \rho_r$ be the probability density functions for red and blue points respectively, $R_{\rho_b},\ R_{\rho_r}$ the cumulative distribution functions, and $R_{\rho_b}^{-1},\ R_{\rho_r}^{-1}$ their inverse functions (usually called "quantile functions" in Statistics)[2]. More precisely,

$$\rho_b(x) = \frac{1}{l_b}\mathbf{1}_{(0,l_b)}(x) \ \Rightarrow\ R_{\rho_b}(x) = \frac{x}{l_b}\cdot \mathbf{1}_{(0,l_b)}(x) \ \Rightarrow\ R_{\rho_b}^{-1}(u) = l_b u \cdot \mathbf{1}_{(0,1)}(u).$$
$$\rho_r(y) = \frac{1}{l_r}\mathbf{1}_{(0,l_r)}(y) \ \Rightarrow\ R_{\rho_r}(y) = \frac{y}{l_r}\cdot \mathbf{1}_{(0,l_r)}(y) \ \Rightarrow\ R_{\rho_r}^{-1}(v) = l_r v \cdot \mathbf{1}_{(0,1)}(v).$$

After re-labeling the n points in order of distance from an endpoint, the probability of the k-th point being in interval $[u, u + \mathrm{d}u]$, denoted by $P_{n,k}(u)$, is

$$P_{n,k}(u)\mathrm{d}u = \frac{n!}{(k-1)!(n-k)!}u^{k-1}(1-u)^{n-k}\mathrm{d}u.$$

Then, for $p \geqslant 2$ even, we can write

$$\begin{aligned}
\mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega_b,\Omega_r}] &= \sum_{k=1}^{n}\int_0^1\int_0^1 P_{n,k}(u)P_{n,k}(v)\left|R_{\rho_b}^{-1}(u) - R_{\rho_r}^{-1}(v)\right|^p \mathrm{d}v\mathrm{d}u \\
&= \sum_{k=1}^{n}\sum_{q=0}^{p}\binom{p}{q}\int_0^1\int_0^1 \frac{n!u^{k-1}(1-u)^{n-k}}{(k-1)!(n-k)!}\,\frac{n!v^{k-1}(1-v)^{n-k}}{(k-1)!(n-k)!}\left(R_{\rho_b}^{-1}(u)\right)^q\left(-R_{\rho_r}^{-1}(v)\right)^{p-q}\mathrm{d}v\mathrm{d}u \\
&= \sum_{k=1}^{n}\sum_{q=0}^{p}\binom{p}{q}\int_0^1\int_0^1 \frac{n!u^{k-1}(1-u)^{n-k}}{(k-1)!(n-k)!}\,\frac{n!v^{k-1}(1-v)^{n-k}}{(k-1)!(n-k)!}(l_b u)^q(-l_r v)^{p-q}\mathrm{d}v\mathrm{d}u \\
&= \sum_{k=1}^{n}\left(\frac{n!}{(k-1)!(n-k)!}\right)^2\sum_{q=0}^{p}\binom{p}{q}(l_b)^q(-l_r)^{p-q}\int_0^1 u^{k-1+q}(1-u)^{n-k}\mathrm{d}u\int_0^1 v^{k-1+p-q}(1-v)^{n-k}\mathrm{d}v
\end{aligned}$$

[2]This proof follows the assumptions and ideas of [1, Lemma 2.6.1]'s proof.

$$
\begin{aligned}
&= \sum_{k=1}^{n} \left(\frac{n!}{(k-1)!(n-k)!} \right)^2 \sum_{q=0}^{p} \binom{p}{q} (l_b)^q (-l_r)^{p-q} \frac{(k-1+q)!(n-k)!}{(n+q)!} \frac{(k-1+p-q)!(n-k)!}{(n+p-q)!} \\
&= \sum_{k=1}^{n} \left(\frac{n!}{(k-1)!} \right)^2 \sum_{q=0}^{p} \binom{p}{q} (l_b)^q (-l_r)^{p-q} \frac{(k-1+q)!}{(n+q)!} \frac{(k-1+p-q)!}{(n+p-q)!} .
\end{aligned}
$$

At $p = 2$, it simplifies to

$$
\begin{aligned}
\mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega_b,\Omega_r}] &= \sum_{k=1}^{n} \left(\frac{n!}{(k-1)!} \right)^2 \sum_{q=0}^{p} \binom{2}{q} (l_b)^q (-l_r)^{2-q} \frac{(k-1+q)!}{(n+q)!} \frac{(k-1+2-q)!}{(n+2-q)!} \\
&= l_r^2 \frac{\sum_{k=1}^{n} k(k+1)}{(n+1)(n+2)} - 2l_b l_r \frac{\sum_{k=1}^{n} k^2}{(n+1)(n+1)} + l_b^2 \frac{\sum_{k=1}^{n} k(k+1)}{(n+1)(n+2)} \\
&= l_r^2 \frac{n(n+1)(2n+1)}{6(n+1)(n+2)} + l_r^2 \frac{n(n+1)}{2(n+1)(n+2)} - 2l_b l_r \frac{n(n+1)(2n+1)}{6(n+1)(n+1)} \\
&\quad + l_b^2 \frac{n(n+1)(2n+1)}{6(n+1)(n+2)} + l_b^2 \frac{n(n+1)}{2(n+1)(n+2)} \\
&= \frac{n}{3} l_r^2 + \frac{n}{3} l_b^2 - \frac{n(2n+1)}{3(n+1)} l_b l_r \ ,
\end{aligned}
$$

which is a homogeneous quadratic polynomial of (l_b, l_r), invariant under $l_b \leftrightarrow l_r$. Upon recalling the standard result at $p = 2$, namely $\mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega,\Omega}] = \frac{1}{3}\frac{n}{n+1}$, eq.(12) follows by simple algebra. □

4.1.2 If Ω_b and Ω_r depend on n

Proposition 3. We assume that $p = 2$. If $0 < l_r < l_b < \frac{n+1}{n} l_r$, then

$$
\mathbb{E}[\mathcal{H}_{\text{opt}}^{[0,l_r],[0,l_r]}] \leqslant \mathbb{E}[\mathcal{H}_{\text{opt}}^{[0,l_b],[0,l_r]}] \leqslant \mathbb{E}[\mathcal{H}_{\text{opt}}^{[0,l_b],[0,l_b]}], \ \forall n \in \mathbb{Z}_+.
$$

If $0 < \frac{n+1}{n} l_r < l_b$, then

$$
\mathbb{E}[\mathcal{H}_{\text{opt}}^{[0,l_b],[0,l_r]}] \geqslant \mathbb{E}[\mathcal{H}_{\text{opt}}^{[0,l_b],[0,l_b]}] \text{ and } \mathbb{E}[\mathcal{H}_{\text{opt}}^{[0,l_b],[0,l_r]}] \geqslant \mathbb{E}[\mathcal{H}_{\text{opt}}^{[0,l_r],[0,l_r]}], \ \forall n \in \mathbb{Z}_+,
$$

Proof of Proposition 3. If $l_r < l_b$, then

$$
((n+1)l_b - nl_r)(l_b - l_r) \geqslant 0 \ \Rightarrow \ (n+1)l_b^2 - (2n+1)l_b l_r + nl_r^2 \geqslant 0 \ \Rightarrow \ l_r^2 \leqslant (n+1)(l_b^2 + l_r^2) - (2n+1)l_b l_r.
$$

If $l_r < l_b < \frac{n+1}{n} l_r$, then

$$
(nl_b - (n+1)l_r)(l_b - l_r) \leqslant 0 \ \Rightarrow \ nl_b^2 - (2n+1)l_b l_r + (n+1)l_r^2 \leqslant 0 \ \Rightarrow \ (n+1)(l_b^2 + l_r^2) - (2n+1)l_b l_r \leqslant l_b^2.
$$

If $l_b \geqslant \frac{n+1}{n} l_r$, then

$$
(nl_b - (n+1)l_r)(l_b - l_r) \geqslant 0 \ \Rightarrow \ nl_b^2 - (2n+1)l_b l_r + (n+1)l_r^2 \geqslant 0 \ \Rightarrow \ (n+1)(l_b^2 + l_r^2) - (2n+1)l_b l_r \geqslant l_b^2.
$$

After noting that

$$
\mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega_b,\Omega_r}] = \big((n+1)(l_b^2 + l_r^2) - (2n+1)l_b l_r\big)\, \mathbb{E}[\mathcal{H}_{\text{opt}}^{[0,1],[0,1]}],
$$

we can finish the proof. □

Let us consider in greater detail the homogeneous quadratic polynomial

$$
P_n(l_b, l_r) \stackrel{\text{def}}{=} \frac{\mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega_b,\Omega_r}]}{\mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega,\Omega}]} = \left[(n+1)(l_b^2 + l_r^2) - (2n+1)l_b l_r\right].
$$

Clearly, $P_n(1,1) = P_n(-1,-1) = 1$, $\forall n \in \mathbb{N}$ (and $l_b = l_r = 1$ and $l_b = l_r = -1$ are the unique solutions independent on n), but if one allows l_b, l_r to depend on n, then the equation $P_n(l_b, l_r) = 1$ has a continuum of non-trivial solutions. Indeed, putting $u = \frac{l_b - l_r}{2}$ and $v = \frac{l_b + l_r}{2}$, it is the ellipse

$$(u, v) \in \mathbb{R}^2 \text{ s.t. } \left(n + \frac{3}{4}\right) u^2 + \frac{v^2}{4} = 1 \, .$$

By the obvious symmetry in the exchange $l_b \leftrightarrow l_r$, we can restrict the discussion to the region $l_b \geq l_r$ (i.e. $u \geq 0$).

Thus, we have the following

Corollary 1. Let $l_b = l_r + \frac{\alpha}{n^\gamma}$ for $\alpha \in \mathbb{R}$ and $\gamma \in \mathbb{R}^+$. We compare $\mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega_r, \Omega_r}]$ and $\mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega_b, \Omega_r}]$.

$$\begin{aligned}
\frac{\mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega_b, \Omega_r}]}{\mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega_r, \Omega_r}]} =& \frac{(n+1)(l_b^2 + l_r^2) - (2n+1) l_b l_r}{l_r^2} \\
=& \frac{(n+1)((l_r + \frac{\alpha}{n^\gamma})^2 + l_r^2) - (2n+1)(l_r + \frac{\alpha}{n^\gamma}) l_r}{l_r^2} \\
=& 1 + \frac{\alpha}{n^\gamma} \frac{1}{l_r} + \frac{\alpha^2}{n^{2\gamma}} \frac{1}{l_r^2} + \frac{\alpha^2}{n^{2\gamma - 1}} \frac{1}{l_r^2}
\end{aligned}$$

Then, as $n \to \infty$,

$$\frac{\mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega_b, \Omega_r}]}{\mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega_r, \Omega_r}]} \underset{n \to \infty}{\sim} \begin{cases} \alpha^2 n^{1-2\gamma} l_r^{-2} & \gamma < \frac{1}{2} \\ 1 + \alpha^2 l_r^{-2} & \gamma = \frac{1}{2} \\ 1 & \gamma > \frac{1}{2} \end{cases} .$$

We remark in particular that the cases $l_b = l_r + \frac{\alpha}{n^\gamma}$ and $l_b = l_r - \frac{\alpha}{n^\gamma}$ give rise to the same result at leading order as $n \to \infty$.

Corollary 1 is a useful asymptotic statement (in the limit $n \to \infty$) for addressing more intricate problems, *e.g.* gluing "one-dimensional" cases (such as the k-star graph in Section 4.3), since it provides a rule of thumb for understanding at sight the leading behavior of $\mathbb{E}\left[\mathcal{H}_{\text{opt}}\right]$ of the problem (i.e. whether the scaling is anomalous or bulk, wrt. to the standard nomenclature). We begin by discussing the upper and lower bounds of the k-star graph in the next part.

4.1.3 On the k-star graph

As an application of Theorem 1, we study now the upper and lower bounds of $\mathbb{E}[\mathcal{H}_{\text{opt}}]$ for the ERAP on the k-star graph presented in [18], that is, $\mathcal{B}$ and $\mathcal{R}$ are uniformly distributed over a tree with k edges of length $\frac{1}{k}$ that share a common vertex O', $k \in \mathbb{N}^*$. In this structure, distances are defined by cases, i.e.,

$$D(B_1, R_1) = \begin{cases} |(D(B_1, O') - D(R_1, O')|, & \text{if } B_1 \text{ and } R_1 \text{ are on the same edge,} \\ D(B_1, O') + D(R_1, O'), & \text{otherwise.} \end{cases}$$

Proposition 4. Let Ω^* be 3-star graph and let the cost function be $D^2(B^{(\Omega^*)}, R^{(\Omega^*)})$. Then we have

$$\frac{4n}{27(n+1)} \leqslant \mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega^*, \Omega^*, n}] \leqslant \frac{n}{3(n+1)}, \ \forall n \geqslant 1.$$

Proof. We use Theorem 1 to prove this. We bound the ERAP on Ω^* above and below using two ERAPs with the same cost function but defined, respectively, on the line segments:

$$\Omega := \Omega_b = \Omega_r = [0, 1], \quad \Omega' := \Omega_b = \Omega_r = \left[0, \frac{2}{3}\right].$$

Now we use the results of Section 4.1.1 for domains of the form $\Omega_b = \Omega_r = [0, l]$ in the case of the 3-star graph. There are 3 choices of 2 arms among 3 available (see Figure 4.2 (a)-(c)). Once this choice is made, the blue $B^{(\Omega)}$ and the red $R^{(\Omega)}$ will fall on some interval of length $\frac{2}{3}$. Notice that case (a), (b) and (c) are not independent: each one carries a probability $\frac{4}{9}$ that that two points are distributed on

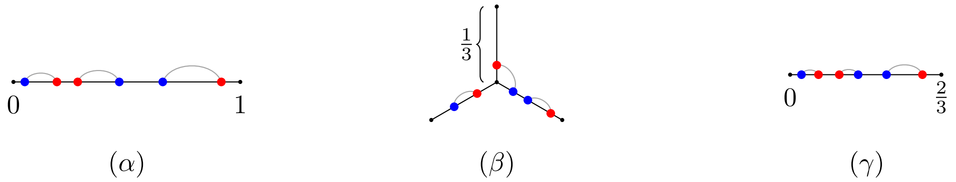

Figure 4.1: (α): $\Omega_b = \Omega_r = [0, 1]$, ($\beta$): Ω_b, Ω_r are 3-star graphs, (γ): $\Omega_b = \Omega_r = [0, \frac{2}{3}]$.

exactly the bold edges (either on the same edge or not). And with reference to Figure 4.2 (d)-(f), it is clear that each case has a probability of occurring with $\frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$. That is, an edge is specified, and both points are distributed on this edge.

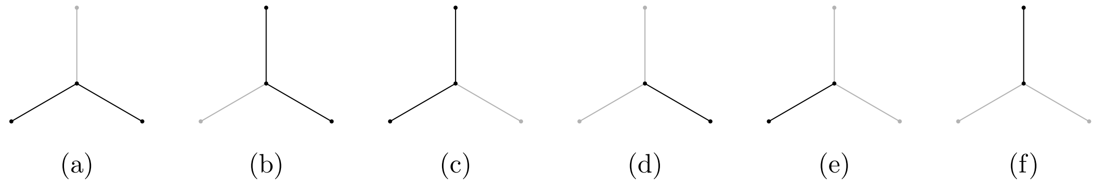

Figure 4.2: 3-star graphs.

Corresponding to Figure 4.1, the cumulative distribution functions are

$$(\alpha):\ \mathbb{P}\left(D\left(B^{(\Omega)}, R^{(\Omega)}\right) \leqslant t\right) = \begin{cases} 0,\ \forall t < 0 \\ -t^2 + 2t,\ \forall t \in [0,1] \\ 1,\ \forall t \geqslant 1 \end{cases},$$

$$\begin{aligned}(\beta):\ \mathbb{P}\left(D\left(B^{(\Omega^*)}, R^{(\Omega^*)}\right) \leqslant t\right) =&\ 3 \times \frac{4}{9}\mathbb{P}\left(D\left(B^{(\Omega')}, R^{(\Omega')}\right) \leqslant t\right) - 3 \times \frac{1}{9}\mathbb{P}\left(D\left(B^{(\overline{\Omega})}, R^{(\overline{\Omega})}\right) \leqslant t\right) \\ =&\ \frac{4}{3}\mathbb{P}\left(D\left(B^{(\Omega)}, R^{(\Omega)}\right) \leqslant \frac{3}{2}t\right) - \frac{1}{3}\mathbb{P}\left(D\left(B^{(\Omega)}, R^{(\Omega)}\right) \leqslant 3t\right),\ \text{where } \overline{\Omega} := \left[0, \frac{1}{3}\right],\end{aligned}$$

and

$$(\gamma):\ \mathbb{P}\left(D\left(B^{(\Omega')}, R^{(\Omega')}\right) \leqslant t\right) = \mathbb{P}\left(D\left(B^{(\Omega)}, R^{(\Omega)}\right) \leqslant \frac{3}{2}t\right).$$

Therefore,

$$D\left(B^{(\Omega')}, R^{(\Omega')}\right) \leq_{\text{st}} D\left(B^{(\Omega^*)}, R^{(\Omega^*)}\right) \leq_{\text{st}} D\left(B^{(\Omega)}, R^{(\Omega)}\right),$$

this means

$$c\left(B^{(\Omega')}, R^{(\Omega')}\right) \leq_{\text{st}} c\left(B^{(\Omega^*)}, R^{(\Omega^*)}\right) \leq_{\text{st}} c\left(B^{(\Omega)}, R^{(\Omega)}\right).$$

By applying Theorem 1 and Proposition 2,

$$\frac{4n}{27(n+1)} = \mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega',\Omega',n}] \leqslant \mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega^*,\Omega^*,n}] \leqslant \mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega,\Omega,n}] = \frac{n}{3(n+1)},\ \forall n \geqslant 1.$$

□

Remark 2. An analogous argument (with minor technical modifications) will give upper and lower bounds for the ERAP on the k-star graph for general exponent $p \geq 1$. The k-star graph also provides an easy situation for illustrating Remark 1-(ii).

Example 2. By reusing the previous notations and by setting $\Omega'' = [0, \frac{2.4}{3}]$, we have

$$\mathbb{P}\left(D\left(B^{(\Omega'')}, R^{(\Omega'')}\right) \leqslant t\right) = \mathbb{P}\left(D\left(B^{(\Omega)}, R^{(\Omega)}\right) \leqslant \frac{3}{2.4}t\right).$$

One instance can show that neither $c\left(B^{(\Omega'')}, R^{(\Omega'')}\right) \leq_{\text{st}} c\left(B^{(\Omega^*)}, R^{(\Omega^*)}\right)$ nor $c\left(B^{(\Omega^*)}, R^{(\Omega^*)}\right) \leq_{\text{st}} c\left(B^{(\Omega'')}, R^{(\Omega'')}\right)$ holds:

$$\mathbb{P}\left(D\left(B^{(\Omega'')}, R^{(\Omega'')}\right) \leqslant \frac{2}{3}\right) = 1 \leqslant \mathbb{P}\left(D\left(B^{(\Omega)}, R^{(\Omega)}\right) \leqslant \frac{2}{3}\right),$$

while,

$$\mathbb{P}\left(D\left(B^{(\Omega'')}, R^{(\Omega'')}\right) \leqslant \frac{1}{4}\right) = \frac{5.4}{10.24} \leqslant \frac{5.12}{10.24} = \frac{1}{2} = \mathbb{P}\left(D\left(B^{(\Omega)}, R^{(\Omega)}\right) \leqslant \frac{1}{4}\right).$$

Using Proposition 2 and Proof of Proposition 4, we get for $n = 1$,

$$\mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega'',\Omega''}] = \frac{5.76}{54}\ ,\ \mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega^*,\Omega^*}] = 3 \times \frac{4}{9}\mathbb{E}(\mathcal{H}_{\text{opt}}^{[0,2/3],[0,2/3]}) - 3 \times \frac{1}{9}\mathbb{E}(\mathcal{H}_{\text{opt}}^{[0,1/3],[0,1/3]}) = \frac{5}{54}\ ,$$

and

$$\lim_{n\to+\infty} \mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega'',\Omega''}] = \frac{5.76}{27}\ .$$

The numerical results in [18] have suggested that

$$\lim_{n\to+\infty} \mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega^*,\Omega^*}] = \frac{7}{27}.$$

Therefore,

$$\lim_{n\to+\infty} \mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega'',\Omega''}] < \lim_{n\to+\infty} \mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega^*,\Omega^*}],\ \text{and for } n=1,\ \mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega'',\Omega''}] > \mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega^*,\Omega^*}].$$

4.2 Applications of Theorem 1 in general dimension $d \geqslant 2$

Theorem 2. Let Ω be a domain with dimension d, and consider one random blue point and one random red point $B_1 = (B_{1,1}, \ldots, B_{1,d})^T \in \Omega, R_1 = (R_{1,1}, \ldots, R_{1,d})^T \in \Omega$. We suppose that $\forall 1 \leqslant i \leqslant d$, $\forall t \geqslant 0$, $\mathbb{P}\left(D\left(B_{1,i}, R_{1,i}\right) \leqslant t\right) = \mathbb{P}\left(D\left(B_{1,1}, R_{1,1}\right) \leqslant t\right)$. Then the 3 following statements hold:

1. For $(\lambda_1, \lambda_2, \ldots, \lambda_d) \in \mathbb{R}^d\backslash\{0\}$, consider a diagonal matrix $\Lambda \in \mathcal{M}_d(\mathbb{R})$ with determinant ± 1,

   $$\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_d)\ ,$$

   which acts as a linear map on B_1 and R_1. Set $B_1' = \Lambda B_1 \in \Omega'$ and $R_1' = \Lambda R_1 \in \Omega'$ for some domain Ω'. Then $\forall p \in \mathbb{R}_+$[3], we prove in Section A.2 that $D\left(B_1, R_1\right) \leq_{\text{st}} D\left(B_1', R_1'\right)$ (and hence by strict monotonicity $D^p\left(B_1, R_1\right) \leq_{\text{st}} D^p\left(B_1', R_1'\right)$, $\forall p \in \mathbb{R}_+$). Therefore, by Theorem 1,

   $$\mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega,\Omega}] \leqslant \mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega',\Omega'}],\ \forall n \geqslant 1. \tag{13}$$

2. Let Λ' and Λ'' be two diagonal $d \times d$-matrices with determinant ± 1,

   $$\Lambda' = \operatorname{diag}(\lambda_1', \lambda_2', \ldots, \lambda_d')\ , \qquad \Lambda'' = \operatorname{diag}(\lambda_1'', \lambda_2'', \ldots, \lambda_d'')\ ,$$

   which transform the points in Ω into the domain Ω' and Ω'', as follows,

   $$R_1' = \Lambda' R_1 \in \Omega',\ B_1' = \Lambda' B_1 \in \Omega',\ R_1'' = \Lambda'' R_1 \in \Omega'',\ B_1'' = \Lambda'' B_1 \in \Omega''.$$

   If $\operatorname{Tr}\left(\operatorname{abs}(\Lambda'')\right) \leqslant \operatorname{Tr}\left(\operatorname{abs}(\Lambda')\right)$, where $\operatorname{abs}(M) := (|M_{ij}|)_{1\leq i,j\leq d}$, then $\forall p \in \mathbb{R}_+$, $D^p\left(B_1'', R_1''\right) \leq_{\text{st}} D^p\left(B_1', R_1'\right)$ and by Theorem 1,

   $$\mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega'',\Omega''}] \leqslant \mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega',\Omega'}],\ \forall n \geqslant 1. \tag{14}$$

[3] For $p \in \mathbb{R}_-$, $D\left(B_1', R_1'\right) \leq_{\text{st}} D\left(B_1, R_1\right)$, $\forall t \geqslant 0$.

3. Let $M \in \mathrm{GL}_d(\mathbb{R})$. We assume that M transforms the points in Ω into the domain $\overline{\Omega}$, thus

$$MR_1 \in \overline{\Omega},\ MB_1 \in \overline{\Omega}.$$

Then $\forall p > 0$, $D^p\left(B_1, R_1\right) \leq_{\mathrm{st}} D^p\left(MB_1, MR_1\right)$ and by Theorem 1,

$$\mathbb{E}[\mathcal{H}_{\mathrm{opt}}^{\Omega,\Omega}] \leqslant \mathbb{E}[\mathcal{H}_{\mathrm{opt}}^{\overline{\Omega},\overline{\Omega}}],\ \forall n \geqslant 1.$$

We now attempt to make Theorem 2 more visible through an example, and to add some comments on this result. The example is comparing the $\mathbb{E}[\mathcal{H}_{\mathrm{opt}}]$ in the ellipses with different eccentricities and equal areas. Unless otherwise specified, we assume in the rest of Section 4.2 that $\mathcal{B}$ ($\mathcal{R}$) are uniformly distributed over Ω_b (Ω_r), and that the cost function is $D^2(x,y)$.

Consider the one-parameter family of ellipses $\mathcal{E}_\lambda = \{(x,y) \in \mathbb{R}^2,\ \text{s.t.}\ (\lambda x)^2 + (\frac{y}{\lambda})^2 \leqslant \frac{1}{\pi},\ \lambda \in (0,1]\}$. Notice that $|\mathcal{E}_\lambda| = 1$, $\forall \lambda$, and that $\mathcal{E}_1 = \mathcal{B}\left((0,0), \frac{1}{\sqrt{\pi}}\right)$ is the unit-area 2-ball. The eccentricity of $\mathcal{E}_\lambda$ is $\sqrt{1-\lambda^4}$. Let $\mathcal{H}_{\mathrm{opt}}^\lambda$ be the ground state energy of the ERAP at $p=2$ on $\mathcal{E}_\lambda$, that is

$$\mathcal{H}_{\mathrm{opt}}^\lambda := \min_{\pi \in \mathcal{S}_n} \sum_{i=1}^{n} D^2(B_i, R_{\pi(i)}),\ \text{with}\ B_i, R_j\ \text{i.i.d.} \sim \mathcal{U}\left(\mathcal{E}_\lambda\right)\text{rvs},\ 1 \leqslant i,j \leqslant n.$$

Remark 3.

(i) In $\mathcal{E}_1$, we get

$$\forall 1 \leqslant i \leqslant d,\ \forall t \geqslant 0,\ \mathbb{P}\left(D\left(B_{1,i}, R_{1,i}\right) \leqslant t\right) = \mathbb{P}\left(D\left(B_{1,1}, R_{1,1}\right) \leqslant t\right).$$

Then, by Theorem 2-1,

$$\mathbb{E}[\mathcal{H}_{\mathrm{opt}}^\lambda] \geqslant \mathbb{E}[\mathcal{H}_{\mathrm{opt}}^1],\ \forall n \geqslant 1, \forall \lambda \in (0,1]. \tag{15}$$

(ii) Theorem 2-2 tells us that, the closer the eccentricity of the boundary is to 0 (when the boundary is a circle), in other words, the closer λ is to 1, the lower $\mathbb{E}[\mathcal{H}_{\mathrm{opt}}^\lambda]$ (see the numerical results in Figure 4.3).

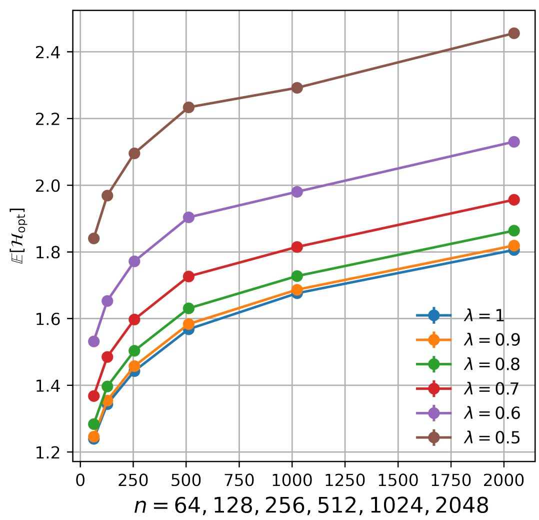

Figure 4.3: $\mathbb{E}[\mathcal{H}_{\mathrm{opt}}^\lambda]$ as a function of n with standard error of the mean (SEM) error bars. The errors are too small to be visible. The number of simulated instances is 1000.

(iii) In the proof of Theorem 2-3, if $M = UIU^T$, I is an identity matrix and $U \in \mathrm{SO}_d(\mathbb{R})$, then $\mathbb{E}[\mathcal{H}_{\mathrm{opt}}^{\Omega,\Omega}] = \mathbb{E}[\mathcal{H}_{\mathrm{opt}}^{\overline{\Omega},\overline{\Omega}}]$, $\forall n \geqslant 1$. It means $\mathbb{E}[\mathcal{H}_{\mathrm{opt}}]$ is the identical in the two congruent regions.

(iv) Further, we observe that for the second part of Theorem 2, if the absolute value of determinant is $a \in \mathbb{R}_+$ instead of 1, the stochastic order holds, so does eq.(14). Therefore, we can compare any two ellipses that are equal in area, but the area is not necessarily equal to 1.

Let us now consider the case when Ω and Ω' are triangles, $\mathcal{R}^{(\Omega)}$, $\mathcal{B}^{(\Omega)}$ are uniformly distributed on Ω, and $\mathcal{R}^{(\Omega')}$, $\mathcal{B}^{(\Omega')}$ are uniformly distributed on Ω'. We define the Cartesian coordinate system so that two vertices of the triangle are located at the same position on the positive half-axis of the horizontal and vertical axes, respectively. This means, their coordinates are $(\alpha, 0)$ and $(0, \alpha)$, with $\alpha \in \mathbb{R}_+$. When the third vertex of the triangle is on the line $y = x$, we get

$$\forall t \geqslant 0, \mathbb{P}\left(D\left(B_{i,x}, R_{i,x}\right) \leqslant t\right) = \mathbb{P}\left(D\left(B_{i,y}, R_{i,y}\right) \leqslant t\right). \tag{16}$$

In this case, Ω is an isosceles triangle (see Figure 4.4 [4] for examples), and we denote the legs as l_1, l_2 and the base as l_3, so $|l_3| = \sqrt{2}\alpha$. We do the transformation $\Lambda = \mathrm{diag}(\lambda, 1/\lambda)$, $\lambda \in \mathbb{R}_+$ on $\mathcal{R}^{(\Omega)}$, $\mathcal{B}^{(\Omega)}$ to get points $\mathcal{R}^{(\Omega')}$, $\mathcal{B}^{(\Omega')}$ (examples of Ω' with $\lambda = \frac{3}{4}, \frac{4}{3}$ are shown in Figure 4.4[5]). Therefore, by using Theorem 2,

$$\mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega,\Omega}] \leqslant \mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega',\Omega'}], \ \forall n \geqslant 1.$$

We then discuss this according to the shape of the triangle:

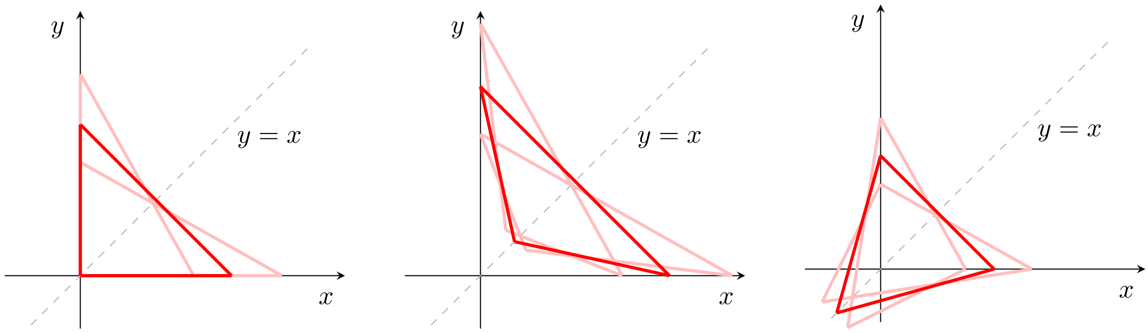

Figure 4.4: Examples of Ω (with red boundary) as right obtuse, or acute triangle. And examples of Ω' (with pink boundary)

1. If Ω is a right triangle, so is Ω'. Ω' is an isosceles triangle only if $\lambda = 1$;
2. if Ω is a obtuse triangle, so is Ω'. Ω' is an isosceles triangle only if $\lambda = 1$;
3. if Ω is an acute triangle, Ω' might be acute, right or obtuse. And except for the case when Ω is an equilateral triangle, there is only one $\lambda > 0$ such that Ω' is an isosceles triangle. The isosceles triangle Ω', which might be acute, right or obtuse (see Figure 4.5[6] [7] for examples), and Ω are not congruent, since the base of Ω gets longer and becomes one leg of Ω'.

In summary, every triangle Ω_1 (including the isosceles triangle) can be obtained by picking a suitable Cartesian coordinate system then transforming from an isosceles triangle Ω_2 with $\lambda > 1$, except for the equilateral triangle. Theorem 2-1 tells us that this transformation leads to a reduction in energy. That is, for every triangle Ω_1 which is not an equilateral triangle, we can find an acute isosceles triangle Ω_2 such that

$$\mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega_2,\Omega_2}] \leqslant \mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega_1,\Omega_1}], \ \forall n \geqslant 1.$$

We can therefore conclude the following.

[4] Coordinates of the vertices of Ω in Figure 4.4 (1)-(3): $\{(2,0),(0,2),(0,0)\}$, $\{(2.5,0),(0,2.5),(0.45,0.45)\}$, $\{(1.5,0),(0,1.5),(-0.58,-0.58)\}$.

[5] Coordinates of the vertices of Ω' in Figure 4.4 (1)-(3): $\{(1.5,0),(0,2.667),(0,0)\}$, $\{(2.667,0),(0,1.5),(0,0)\}$; $\{(1.875,0),(0,3.333),(0.3375,0.6)\}$, $\{(3.333,0),(0,1.875),(0.6,0.3375)\}$; $\{(1.125,0),(0,2),(-0.435,-0.773)\}$, $\{(2,0),(0,1.125),(-0.773,-0.435)\}$.

[6] Coordinates of the vertices of Ω in Figure 4.5 (1)-(3): $\{(1.9,0),(0,1.9),(-0.1026,-0.1026)\}$, $\{(1.3375,0),(0,1.3375),(-0.8266,-0.8266)\}$, $\{(1.2,0),(0,1.2),(-1.067,-1.067)\}$.

[7] Coordinates of the vertices of Ω' in Figure 4.5 (1)-(3): $\{(1.0973,0),(0,3.29),(-0.0593,-0.1777)\}$, $\{(1.7013,0),(0,1.0515),(-1.0515,-0.6498)\}$, $\{(2.2458,0),(0,0.6412),(-1.997,-0.5701)\}$.

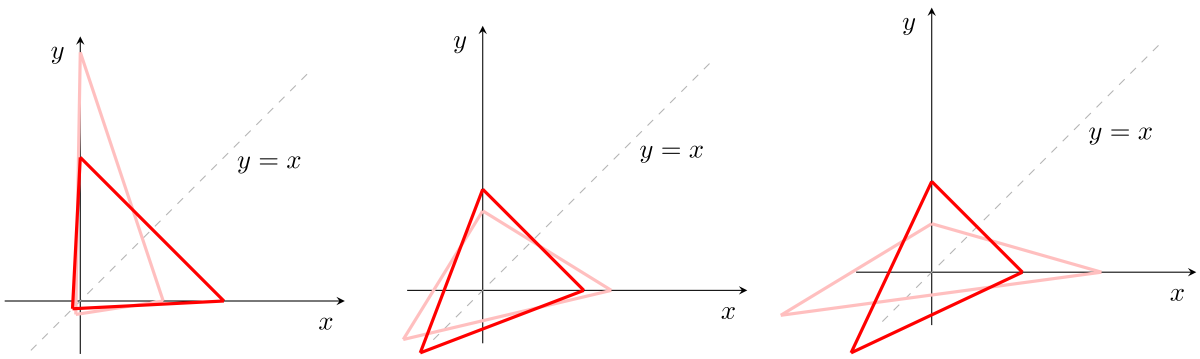

Figure 4.5: Examples of Ω (with red boundary) as acute triangles and Ω' as acute, right or obtuse triangles (with pink boundary)

Proposition 5. If Ω is an equilateral triangle and Ω' is a triangle with the same area, we always have

$$\mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega,\Omega}] \leqslant \mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega',\Omega'}], \ \forall n \geqslant 1. \tag{17}$$

To verify Proposition 5, we only need to compare $\mathbb{E}[\mathcal{H}_{\text{opt}}]$ over isosceles triangles for $n = 1$ (see Lemma 1).

Lemma 1. Let the family of isosceles triangles be

$$\Lambda\Omega = \Omega^{(\lambda)} = \left\{ (x, y) \in \mathbb{R}^2 \text{ s.t. } \frac{1}{\lambda^2}(x-\lambda) \leq y \leq \frac{1}{\lambda^2}(\lambda - x), \ x \in [0, \lambda], \ \lambda > 0 \right\}.$$

Then, we have, for $n = 1$,

$$\mathbb{E}[\mathcal{H}_{\text{opt}}^{\lambda}] \geq \mathbb{E}[\mathcal{H}_{\text{opt}}^{3^{1/4}}] \, . \tag{18}$$

Proof. Remark that the uniform distribution over $\Omega^{(1)}$ has the following probability density function

$$\rho_{X,Y}(x, y) = \mathbf{1}_{\Omega}(x, y)$$

and hence the following marginals

$$\begin{aligned} \rho_X(x) &= 2(1-x)\mathbf{1}_{[0,1]}(x), \\ \rho_Y(y) &= (1+y)\mathbf{1}_{[-1,0]}(y) + (1-y)\mathbf{1}_{[0,1]}(y). \end{aligned} \tag{19}$$

For general λ, the marginals read

$$\begin{aligned} \rho_X^{(\lambda)}(x) &= \frac{2}{\lambda^2}(\lambda - x)\mathbf{1}_{[0,\lambda]}(x), \\ \rho_Y^{(\lambda)}(y) &= \lambda(1+\lambda y)\mathbf{1}_{[-\frac{1}{\lambda},0]}(y) + \lambda(1-\lambda y)\mathbf{1}_{[0,\frac{1}{\lambda}]}(y) \, , \end{aligned} \tag{20}$$

which recover eq.(19) at $\lambda = 1$. Now, for $s \in \mathbb{N}$, the moments write

$$\begin{aligned} \mathbb{E}\left[X^s\right] &= 2\frac{\lambda^s}{(s+2)(s+1)}, \ s \in \mathbb{N}, \\ \mathbb{E}\left[Y^s\right] &= \begin{cases} 2\frac{\left(\frac{1}{\lambda}\right)^s}{(s+2)(s+1)} & s \text{ even} \\ 0 & s \text{ odd} \, . \end{cases} \end{aligned} \tag{21}$$

As $B_x, R_x \sim X$; $B_y, R_y \sim Y$ and $B \perp\!\!\!\perp R$,

$$\begin{aligned} \mathbb{E}[\mathcal{H}_{\text{opt}}^{\lambda}] =& \ \mathbb{E}\left[(B_x - R_x)^2 + (B_y - R_y)^2\right] = \mathbb{E}\left[(B_x - R_x)^2\right] + \mathbb{E}\left[(B_y - R_y)^2\right] \\ =& \ 2\mathbb{E}\left[X^2\right] - 2\mathbb{E}\left[X\right]^2 + 2\mathbb{E}\left[Y^2\right] - 2\mathbb{E}\left[Y\right]^2 = 2\left(\mathbb{E}\left[X^2\right] - \mathbb{E}\left[X\right]^2 + \mathbb{E}\left[Y^2\right]\right) \\ =& \ 2\left(\frac{\lambda^2}{6} - \left(\frac{\lambda}{3}\right)^2 + \frac{\frac{1}{\lambda^2}}{6}\right) = \frac{\lambda^2}{9} + \frac{1}{3\lambda^2} \ . \end{aligned} \tag{22}$$

Then, when $\lambda = 3^{1/4}$, eq.(22) reaches the minimum. Thus, eq.(18) is proven, and $\Omega^{(3^{1/4})}$ is an equilateral triangle. $\square$

Remark 4. Similarly, we can extend Proposition 5 to n-sided convex polygons, i.e., if Ω is an n-sided regular convex polygon, and Ω' is an n-sided convex polygon with the same area, then the eq.(17) holds.

5 Research perspectives

The purpose of the current study was to understand several specific problems mentioned in the first section. In Section 3 we introduced a theorem that can be used to study these different problems. However, we have not discussed each situation fully. At the end of this report, we propose some works that need to be perfected and present some research conjectures.

A number of limitations needs to be noted regarding Theorem 1. First, as mentioned in the remark 1 (ii) in Section 3, we have not proved the necessity of the stochastic order of cost functions for getting eq.(7). We believe that further study of this condition may contribute to our understanding of the Euclidean Random Assignment Problems in general. Second, it would be interesting to study the sufficient conditions which are more easily dealt with, as calculating the cumulative distribution function of $c(B_1, R_1)$ might be challenging (analytically), particularly for the higher dimensional problems.

Therefore, we propose an improvement of Theorem 1, using second-order stochastic dominance to compare different Euclidean Random Assignment Problems.

Conjecture 1. Under the same assumptions as in Theorem 1,

$$c\left(B^{(\Omega_3)}, R^{(\Omega_4)}\right) \leq_{\text{ssd}} c\left(B^{(\Omega_3)}, R^{(\Omega_4)}\right), \tag{23}$$

if and only if

$$\mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega_3,\Omega_4}] \leqslant \mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega_1,\Omega_2}],\ \forall n \geqslant 1. \tag{24}$$

In spite of its limitations, this theorem offers valuable insights into studying some difficult cases. In terms of future work, we are considering several applications. Firstly, it is possible to find upper and lower bounds for $\mathbb{E}[\mathcal{H}_{\text{opt}}]$ on the k-star graphs that are closer to the exact solution using a similar method (with some refinement) to Proposition 4. Secondly, for the comparison of $\mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega,\Omega}]$ and $\mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega,\Omega'}]$, the issue when Ω' depends on n, as we show in Section 1.3, is an intriguing one which could be usefully explored in further research. In particular, we have the following conjectures as $n \to \infty$.

Conjecture 2. Based on the same assumptions as in Theorem 1,

$$\lim_{n\to+\infty} \mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega_3,\Omega_4}] \leqslant \lim_{n\to+\infty} \mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega_1,\Omega_2}]. \tag{25}$$

if and only if

$$\int_{-\infty}^{t} \mathbb{P}(c\left(B^{(\Omega_3)}, R^{(\Omega_4)}\right) \leqslant t)\mathrm{d}t \geqslant \int_{-\infty}^{t} \mathbb{P}(c\left(B^{(\Omega_1)}, R^{(\Omega_2)}\right) \leqslant t)\mathrm{d}t,\ \text{for } t \to 0. \tag{26}$$

Conjecture 3. Let $\Omega' \subset \Omega$, and $\max\{D(y, \partial\Omega),\ y \in \partial\Omega'\} \leqslant \frac{1}{\sqrt{n}}$. Then

$$\lim_{n\to+\infty} \mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega,\Omega'}] \geqslant \lim_{n\to+\infty} \mathbb{E}[\mathcal{H}_{\text{opt}}^{(\Omega,\Omega)}].$$

The proof of Conjecture 3 for one case (as shown in Figure B.1) :

$$\Omega' = [0,1] \times [0, 1 - \frac{1}{\sqrt{n}}] \subset [0,1]^2 = \Omega$$

is provided in Section A.3. The proof of the general case is an open question, and the link between ERAP and Monge-Kantorovich problem reviewed in section 2 might be helpful in finding a rigorous proof.

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A Proofs

A.1 Proof of Theorem 1

Let $\{X_i, 1 \leqslant i \leqslant m\}$ be m r.v.s, and $\{Y_i, 1 \leqslant i \leqslant m\}$ be m r.v.s such that

$$F_{X_i}(t) \geqslant F_{Y_i}(t),\ \forall t \geqslant 0,\ \forall 1 \leqslant i \leqslant m. \tag{27}$$

We will discuss some findings about $\{X_i\}$ and $\{Y_i\}$ in **Steps 1** and **2**. Then applying these findings, by substitution in **Steps 3** and **4**, we will get eq.(8) and eq.(9). In other words, we will obtain the stochastic order of $\min\limits_{\pi \in S_n} \left\{ \sum\limits_{i=1}^{n} c(B_i^{(\Omega_1)}, R_{\pi(i)}^{(\Omega_2)}) \right\}$ and $\min\limits_{\pi \in S_n} \left\{ \sum\limits_{i=1}^{n} c(B_i^{(\Omega_3)}, R_{\pi(i)}^{(\Omega_4)}) \right\}$. In **Step 5**, we will finish the proof.

Step 1. We now prove that if $\{X_i\}$ are i.i.d. r.v.s, $\{Y_i\}$ as well, then $\forall 1 \leqslant k \leqslant m$,

$$F_{\sum_{j=1}^k X_j}(t) \geqslant F_{\sum_{j=1}^k Y_j}(t),\ \forall t \geqslant 0. \tag{28}$$

In fact, $\forall t \geqslant 0$,

$$F_{X_i}(t) \geqslant F_{Y_i}(t)\ \Rightarrow\ \mathbb{P}(X_i \leqslant t) \geqslant \mathbb{P}(Y_i \leqslant t). \tag{29}$$

By independence, we have

$$F_{\sum_{j=1}^k X_j}(t) = \mathbb{P}\left(\sum_{j=1}^{k} X_j \leqslant t\right) = \prod_{j=1}^{k} \mathbb{P}\left(X_j \leqslant t - \sum_{i=1}^{j-1} X_i\right),\ \forall 1 \leqslant k \leqslant m. \tag{30}$$

or equivalently

$$F_{\sum_{j=1}^k X_j}(t) = \mathbb{P}\left(\sum_{j=1}^{k} X_j \leqslant t\right) = \prod_{j=1}^{k} \mathbb{P}\left(X_j \leqslant t_j\right),\ \forall\ 0 \leqslant t_j \leqslant t,\ \forall 1 \leqslant k \leqslant m \text{ with } \sum_{j=1}^{k} t_j = t\,.$$

An analogous expression holds for the cdf of $\sum_{j=1}^k Y_j$'s, namely,

$$F_{\sum_{j=1}^k Y_j}(t) = \mathbb{P}\left(\sum_{j=1}^{k} Y_j \leqslant t\right) = \prod_{j=1}^{k} \mathbb{P}\left(Y_j \leqslant t - \sum_{i=1}^{j-1} Y_i\right),\ \forall 1 \leqslant k \leqslant m. \tag{31}$$

Therefore, we apply eq.(29) to the right-hand side of eq.(30) and eq.(31) to get eq.(28).

Step 2. Now we prove the following inequality by contradiction,

$$F_{\min_{1 \leqslant j \leqslant m} \{X_j\}}(t) \geqslant F_{\min_{1 \leqslant j \leqslant m} \{Y_j\}}(t),\ \forall t \geqslant 0. \tag{32}$$

If there exist $t \geqslant 0$ and $1 \leqslant i_1, i_2 \leqslant m$ such that

$$F_{\min_{1 \leqslant j \leqslant m}\{X_j\}}(t) = F_{X_{i_1}}(t),\ F_{\min_{1 \leqslant j \leqslant m}\{Y_j\}}(t) = F_{Y_{i_2}}(t),\ \text{and } F_{X_{i_1}}(t) < F_{Y_{i_2}}(t),$$

then

$$F_{X_{i_1}}(t) < F_{Y_{i_2}}(t) \leqslant F_{Y_{i_1}}(t),$$

which reaches a contradiction.

Step 3. Due to the i.i.d. assumption,

$$F_{c\left(B_i^{(\Omega_1)},R_j^{(\Omega_2)}\right)}(t) = F_{c\left(B^{(\Omega_1)},R^{(\Omega_2)}\right)}(t),\ \forall 1 \leqslant i,j \leqslant n,\ \forall t \geqslant 0.$$

It means, $\forall 0 \leqslant i,j \leqslant n,\ c\left(B_i^{(\Omega_1)}, R_j^{(\Omega_2)}\right)$ follow the identical distributions. We remark that, for every $\pi, \pi' \in S_n$,

$$\left\{c\left(B_i^{(\Omega_1)}, R_{\pi(j)}^{(\Omega_2)}\right),\ \forall 1 \leqslant i \leqslant n\right\} \text{ and } \left\{c\left(B_i^{(\Omega_3)}, R_{\pi'(j)}^{(\Omega_4)}\right),\ \forall 1 \leqslant i \leqslant n\right\}$$

are sequences of n i.i.d. r.v.s. By the assumption (eq.(6)), $\forall 1 \leqslant i,j \leqslant n$,

$$F_{c\left(B_i^{(\Omega_3)},R_{\pi(i)}^{(\Omega_4)}\right)}(t) \geqslant F_{c\left(B_j^{(\Omega_1)},R_{\pi'(j)}^{(\Omega_2)}\right)}(t),\ \forall t \geqslant 0\ ,$$

and we bring the following substitutions into eq.(27) and eq.(28) of **Step 1**

$$\begin{aligned} m &\longrightarrow n, \\ k &\longrightarrow n, \\ X_j &\longrightarrow c(B_i^{(\Omega_3)}, R_{\pi(i)}^{(\Omega_4)}), \\ Y_j &\longrightarrow c(B_i^{(\Omega_1)}, R_{\pi'(i)}^{(\Omega_2)}), \end{aligned}$$

then,

$$F_{\sum_{i=1}^n c(B_i^{(\Omega_3)},R_{\pi(i)}^{(\Omega_4)})}(t) \geqslant F_{\sum_{i=1}^n c(B_i^{(\Omega_1)},R_{\pi'(i)}^{(\Omega_2)})}(t),\ \forall t \geqslant 0. \tag{33}$$

Step 4. We remark that

$$\left\{\sum_{i=1}^n c(B_i^{(\Omega_1)}, R_{\pi(i)}^{(\Omega_2)}),\ \forall \pi \in S_n\right\}$$

are $n!$ identically distributed r.v.s because $\forall 0 \leqslant i,j \leqslant n,\ c\left(B_i^{(\Omega_1)}, R_j^{(\Omega_2)}\right)$ follow the identical distributions. By the same reason,

$$\left\{\sum_{i=1}^n c(B_i^{(\Omega_3)}, R_{\pi(i)}^{(\Omega_4)}),\ \forall \pi \in S_n\right\}$$

are $n!$ identically distributed r.v.s. Using eq.(33), and we bring the following substitutions into eq.(27) and eq.(32) of **Step 2**

$$\begin{aligned} m &\longrightarrow n!, \\ \{X_j\} &\longrightarrow \left\{\sum_{i=1}^n c(B_i^{(\Omega_3)}, R_{\pi(i)}^{(\Omega_4)}), \forall \pi \in S_n\right\}, \\ \{Y_j\} &\longrightarrow \left\{\sum_{i=1}^n c(B_i^{(\Omega_1)}, R_{\pi(i)}^{(\Omega_2)}), \forall \pi \in S_n\right\}, \end{aligned}$$

then,

$$F_{\min\left\{\sum_{i=1}^n c(B_i^{(\Omega_3)},R_{\pi(i)}^{(\Omega_4)}),\forall\pi\in S_n\right\}}(t) \geqslant F_{\min\left\{\sum_{i=1}^n c(B_i^{(\Omega_1)},R_{\pi(i)}^{(\Omega_2)}),\forall\pi\in S_n\right\}}(t),\ \forall t \geqslant 0. \tag{34}$$

Step 5. Since $\min\limits_{\pi\in S_n} \sum\limits_{i=1}^n c(x,y)$ is non-negative, $\mathbb{E}\left[\min\limits_{\pi\in S_n} \sum\limits_{i=1}^n c(x,y)\right] = \int_0^{+\infty}\left(1 - F_{\min\limits_{\pi\in S_n} \sum\limits_{i=1}^n c(x,y)}(t)\right)\mathrm{d}t.$

Thus,

$$
\begin{aligned}
&\mathbb{E}\left[\min_{\pi\in S_n}\sum_{i=1}^{n} c(B_i^{(\Omega_1)}, R_{\pi(i)}^{(\Omega_2)})\right] - \mathbb{E}\left[\min_{\pi\in S_n}\sum_{i=1}^{n} c(B_i^{(\Omega_3)}, R_{\pi(i)}^{(\Omega_4)})\right] \\
&= \int_0^{+\infty}\left(1 - F_{\min\limits_{\pi\in S_n}\sum\limits_{i=1}^{n} c(B_i^{(\Omega_1)}, R_{\pi(i)}^{(\Omega_2)})}(t)\right)\mathrm{d}t - \int_0^{+\infty}\left(1 - F_{\min\limits_{\pi\in S_n}\sum\limits_{i=1}^{n} c(B_i^{(\Omega_3)}, R_{\pi(i)}^{(\Omega_4)})}(t)\right)\mathrm{d}t \\
&= \int_0^{+\infty}\left(F_{\min\limits_{\pi\in S_n}\sum\limits_{i=1}^{n} c(B_i^{(\Omega_3)}, R_{\pi(i)}^{(\Omega_4)})}(t) - F_{\min\limits_{\pi\in S_n}\sum\limits_{i=1}^{n} c(B_i^{(\Omega_1)}, R_{\pi(i)}^{(\Omega_2)})}(t)\right)\mathrm{d}t \geqslant 0\ .
\end{aligned}
$$

We conclude that

$$
\mathbb{E}\left[\min_{\pi\in S_n}\sum_{i=1}^{n} c(B_i^{(\Omega_1)}, R_{\pi(i)}^{(\Omega_2)})\right] \geqslant \mathbb{E}\left[\min_{\pi\in S_n}\sum_{i=1}^{n} c(B_i^{(\Omega_3)}, R_{\pi(i)}^{(\Omega_4)})\right],\ \forall n \geqslant 1. \tag{35}
$$

We notice that if

$$
\int_0^{+\infty}\left(1 - F_{\min\limits_{\pi\in S_n}\sum\limits_{i=1}^{n} c(B_i^{(\Omega_1)}, R_{\pi(i)}^{(\Omega_2)})}(t)\right)\mathrm{d}t = +\infty,\ \text{or,}\quad \int_0^{+\infty}\left(1 - F_{\min\limits_{\pi\in S_n}\sum\limits_{i=1}^{n} c(B_i^{(\Omega_3)}, R_{\pi(i)}^{(\Omega_4)})}(t)\right)\mathrm{d}t = +\infty,
$$

eq.(35) holds naturally.

A.2 Proof of Theorem 2

1. By assumption, $\forall 1 \leqslant i \leqslant d,\ \forall t \geqslant 0$, $\mathbb{P}\left(D^p\left(B_{1,i}, R_{1,i}\right) = t\right) = \mathbb{P}\left(D^p\left(B_{1,1}, R_{1,1}\right) = t\right)$. Thus,

$$\begin{aligned}
& \mathbb{P}\left(|\lambda_1|^p|B_{1,1} - R_{1,1}|^p + |\lambda_2|^p|B_{1,2} - R_{1,2}|^p + \cdots + |\lambda_d|^p|B_{1,d} - R_{1,d}|^p \leqslant t\right) \\
= \ & \mathbb{P}\left(|\lambda_1|^p|B_{1,2} - R_{1,2}|^p + |\lambda_2|^p|B_{1,3} - R_{1,3}|^p + \cdots + |\lambda_d|^p|B_{1,1} - R_{1,1}|^p \leqslant t\right) \\
= \ & \ldots \\
= \ & \mathbb{P}\left(|\lambda_1|^p|B_{1,d} - R_{1,d}|^p + |\lambda_2|^p|B_{1,1} - R_{1,1}|^p + \cdots + |\lambda_d|^p|B_{1,d-1} - R_{1,d-1}|^p \leqslant t\right) .
\end{aligned}$$

We remark that, if for every $t \leqslant 0$, $\mathbb{P}(X \leqslant t) = \mathbb{P}(Y \leqslant t)$ then $\mathbb{P}(X \leqslant t) = \mathbb{P}(X + Y \leqslant 2t)$. Therefore,

$$\begin{aligned}
& \mathbb{P}\left(|\lambda_1|^p|B_{1,1} - R_{1,1}|^p + |\lambda_2|^p|B_{1,2} - R_{1,2}|^p + \cdots + |\lambda_d|^p|B_{1,d} - R_{1,d}|^p \leqslant t\right) \\
=& \mathbb{P}\left(\sum_{i=1}^{d}|\lambda_i|^p|B_{1,1} - R_{1,1}|^p + \sum_{i=1}^{d}|\lambda_i|^p|B_{1,2} - R_{1,2}|^p + \cdots + \sum_{i=1}^{d}|\lambda_i|^p|B_{1,d} - R_{1,d}|^p \leqslant d \times t\right) \\
=& \mathbb{P}\left(\frac{\sum_{i=1}^{d}|\lambda_i|^p}{d}|B_{1,1} - R_{1,1}|^p + \frac{\sum_{i=1}^{d}|\lambda_i|^p}{d}|B_{1,2} - R_{1,2}|^p + \cdots + \frac{\sum_{i=1}^{d}|\lambda_i|^p}{d}|B_{1,d} - R_{1,d}|^p \leqslant t\right).
\end{aligned}$$

Meanwhile,

$$\begin{aligned}
& \mathbb{P}\left(|B_{1,1} - R_{1,1}|^p + |B_{1,2} - R_{1,2}|^p + \cdots + |B_{1,d} - R_{1,d}|^p \leqslant t\right) \\
=& \mathbb{P}\left(\frac{\sum_{i=1}^{d}|\lambda_i|^p}{d}|B_{1,1} - R_{1,1}|^p + \frac{\sum_{i=1}^{d}|\lambda_i|^p}{d}|B_{1,2} - R_{1,2}|^p + \cdots + \frac{\sum_{i=1}^{d}|\lambda_i|^p}{d}|B_{1,d} - R_{1,d}|^p \leqslant \frac{\sum_{i=1}^{d}|\lambda_i|^p}{d}t\right).
\end{aligned}$$

As $\left|\prod_{i=1}^{d}\lambda_i\right| = 1$, by the inequality of arithmetic and geometric means, we get

$$\frac{\sum_{i=1}^{d}|\lambda_i|^p}{d} \geqslant \left(\prod_{i=1}^{d}|\lambda_i|^p\right)^{1/d} = \left(\prod_{i=1}^{d}|\lambda_i|\right)^{p/d} = 1.$$

In conclusion,

$$\begin{aligned}
F_{D^p\left(B_1', R_1'\right)}(t) =\ & \mathbb{P}\left(D^p\left(B_1', R_1'\right) \leqslant t\right) = \mathbb{P}\left(\mathrm{Tr}\left((\mathrm{abs}(\Lambda B_1 - \Lambda R_1))^p\right) \leqslant t\right) \\
=\ & \mathbb{P}\left(|\lambda_1|^p|B_{1,1} - R_{1,1}|^p + |\lambda_2|^p|B_{1,2} - R_{1,2}|^p + \cdots + |\lambda_d|^p|B_{1,d} - R_{1,d}|^p \leqslant t\right) \\
\leqslant\ & \mathbb{P}\left(|B_{1,1} - R_{1,1}|^p + |B_{1,2} - R_{1,2}|^p + \cdots + |B_{1,d} - R_{1,d}|^p \leqslant t\right) \\
=\ & \mathbb{P}\left(\mathrm{Tr}\left((\mathrm{abs}(B_1 - R_1))^p\right) \leqslant t\right) = \mathbb{P}\left(D^p\left(B_1, R_1\right) \leqslant t\right) = F_{D^p(B_1, R_1)}(t),\ \forall t \geqslant 0,
\end{aligned}$$

i.e.,

$$D^p\left(B_1, R_1\right) \leq_{\mathrm{st}} D^p\left(B_1', R_1'\right),\ \forall p \in \mathbb{R}_+,$$

and hence, by Theorem 1,

$$\mathbb{E}[\mathcal{H}_{\mathrm{opt}}^{\Omega,\Omega}] \leqslant \mathbb{E}[\mathcal{H}_{\mathrm{opt}}^{\Omega',\Omega'}],\ \forall n \geqslant 1.$$

2. We use monotonicity of trace functions (see *e.g.* [41, Section 2.2]) w.r.t. the function $f(t) = t^p$ (which is monotone increasing $\forall p > 0$): For any Hermitian matrix A, the function $A \to \mathrm{Tr}[A^p]$ is monotone

increasing on $\mathbb{R}_+$. In our case,

$$\operatorname{Tr}\left(\operatorname{abs}(\Lambda'')\right) \leqslant \operatorname{Tr}\left(\operatorname{abs}(\Lambda')\right) \implies \operatorname{Tr}\left((\operatorname{abs}(\Lambda''))^p\right) \leqslant \operatorname{Tr}\left((\operatorname{abs}(\Lambda'))^p\right) \iff \sum_{i=1}^{d} |\lambda_i''|^p \leqslant \sum_{i=1}^{d} |\lambda_i'|^p.$$

Then, as above, by the assumption, $\forall 1 \leqslant i \leqslant d$, $\forall t \geqslant 0$, $\mathbb{P}\left(D^p\left(B_{1,i}, R_{1,i}\right) = t\right) = \mathbb{P}\left(D^p\left(B_{1,1}, R_{1,1}\right) = t\right)$, we have

$$\begin{aligned}
F_{D^p\left(B_1'', R_1''\right)}(t) &= \mathbb{P}\left(D^p\left(B_1'', R_1''\right) \leqslant t\right) = \mathbb{P}\left(\operatorname{Tr}\left((\operatorname{abs}(\Lambda'' B_1 - \Lambda'' R_1))^p\right) \leqslant t\right) \\
&= \mathbb{P}\left(|\lambda_1''|^p |B_{1,1} - R_{1,1}|^p + |\lambda_2''|^p |B_{1,2} - R_{1,2}|^p + \cdots + |\lambda_d''|^p |B_{1,d} - R_{1,d}|^p \leqslant t\right) \\
&= \mathbb{P}\left(\frac{\sum_{i=1}^{d} |\lambda_i''|^p}{d} |B_{1,1} - R_{1,1}|^p + \frac{\sum_{i=1}^{d} |\lambda_i''|^p}{d} |B_{1,2} - R_{1,2}|^p + \cdots + \frac{\sum_{i=1}^{d} |\lambda_i''|^p}{d} |B_{1,d} - R_{1,d}|^p \leqslant t\right) \\
&= \mathbb{P}\left(|B_{1,1} - R_{1,1}|^p + |B_{1,2} - R_{1,2}|^p + \cdots + |B_{1,d} - R_{1,d}|^p \leqslant \frac{d}{\sum_{i=1}^{d} |\lambda_i''|^p} \times t\right) \\
&\geqslant \mathbb{P}\left(|B_{1,1} - R_{1,1}|^p + |B_{1,2} - R_{1,2}|^p + \cdots + |B_{1,d} - R_{1,d}|^p \leqslant \frac{d}{\sum_{i=1}^{d} |\lambda_i'|^p} \times t\right), \text{ since } \sum_{i=1}^{d} |\lambda_i''|^p \leqslant \sum_{i=1}^{d} |\lambda_i'|^p \\
&= \mathbb{P}\left(\frac{\sum_{i=1}^{d} |\lambda_i'|^p}{d} |B_{1,1} - R_{1,1}|^p + \frac{\sum_{i=1}^{d} |\lambda_i'|^p}{d} |B_{1,2} - R_{1,2}|^p + \cdots + \frac{\sum_{i=1}^{d} |\lambda_i'|^p}{d} |B_{1,d} - R_{1,d}|^p \leqslant t\right) \\
&= \mathbb{P}\left(|\lambda_1'|^p |B_{1,1} - R_{1,1}|^p + |\lambda_2'|^p |B_{1,2} - R_{1,2}|^p + \cdots + |\lambda_d'|^p |B_{1,d} - R_{1,d}|^p \leqslant t\right) \\
&= \mathbb{P}\left(\operatorname{Tr}\left((\operatorname{abs}(\Lambda' B_1 - \Lambda' R_1))^p\right) \leqslant t\right) = \mathbb{P}\left(D^p\left(B_1', R_1'\right) \leqslant t\right) = F_{D^p\left(B_1', R_1'\right)}(t), \ \forall t \geqslant 0.
\end{aligned}$$

Therefore, $D^p\left(B_1'', R_1''\right) \leq_{\mathrm{st}} D^p\left(B_1', R_1'\right)$ and using again Theorem 1,

$$\mathbb{E}[\mathcal{H}_{\mathrm{opt}}^{\Omega'', \Omega''}] \leqslant \mathbb{E}[\mathcal{H}_{\mathrm{opt}}^{\Omega', \Omega'}], \ \forall n \geqslant 1.$$

3. Since M is real symmetric, it is diagonalizable by rotation matrices: $M = U \Lambda U^T$, where
   (a) Λ is a diagonal $d \times d$ matrix, and $|\det(\Lambda)| = 1$, since $|\det(M)| = 1$;
   (b) $U \in \mathrm{SO}_d(\mathbb{R})$.

   Clearly

$$D\left(U \Lambda U^T B_1, U \Lambda U^T R_1\right) = D\left(\Lambda U^T B_1, \Lambda U^T R_1\right), \qquad D\left(U^T B_1, U^T R_1\right) = D\left(B_1, R_1\right),$$

   so that, $\forall p$, we have

$$F_{D^p(U \Lambda U^T B_1, U \Lambda U^T R_1)}(t) = F_{D^p(\Lambda U^T B_1, \Lambda U^T R_1)}(t), \qquad F_{D^p(U^T B_1, U^T R_1)}(t) = F_{D^p(B_1, R_1)}(t), \ \forall t \geqslant 0.$$

   According to Theorem 2-1, $\forall p > 0$,

$$F_{D^p(\Lambda U^T B_1, \Lambda U^T R_1)}(t) \leqslant F_{D^p(U^T B_1, U^T R_1)}(t), \ \forall t \geqslant 0.$$

In summary, $\forall p>0$,

$$F_{D^p(MB_1,MR_1)}(t)=F_{D^p(U\Lambda U^T B_1,U\Lambda U^T R_1)}(t)\leqslant F_{D^p(B_1,R_1)}(t),\ \forall t\geqslant 0,$$

i.e. $D^p\left(B_1,R_1\right)\leq_{\mathrm{st}} D^p\left(MB_1,MR_1\right)$. Therefore, after using Theorem 1,

$$\mathbb{E}[\mathcal{H}_{\mathrm{opt}}^{\Omega,\Omega}]\leqslant\mathbb{E}[\mathcal{H}_{\mathrm{opt}}^{\overline{\Omega},\overline{\Omega}}],\ \forall n\geqslant 1.$$

A.3 Proof of Proposition 3

For case 1: $\Omega = [0,1]^2$ and $\Omega' = [0,1] \times [0, 1 - \frac{1}{\sqrt{n}}]$.

Let us consider $B_i, R_j \in \Omega = [0,1]^2$, for $i, j = 1, \dots, n$. We consider a set Ω' which is image of Ω via a linear transformation, $\Omega' = \Lambda\Omega$, where $\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 1 - \frac{1}{\sqrt{n}} \end{pmatrix}$. That is, letting $R_j = (R_{j,x}, R_{j,y})^T$ (and similarly $B_j = (B_{j,x}, B_{j,y})^T$), we transform only the red points and consider the ERAP between $R'_j = \Lambda R_j \in \Omega' = [0,1] \times [0, 1 - \frac{1}{\sqrt{n}}]$ and $B_i = (B_{i,x}, B_{i,y})^T \in \Omega$. We have

$$
\begin{aligned}
\mathcal{H}_{\text{opt}}^{\Omega,\Omega'} &= \min_{\pi \in \mathcal{S}_n} \sum_{i=1}^n D^2(B_i, R'_{\pi(i)}) \\
&= \sum_{i=1}^n D^2(B_i, R'_{\hat{\pi}(i)}) \qquad \left(\text{here, we denote } \hat{\pi}' \text{ such that } \sum_{i=1}^n D^2(B_i, R'_{\hat{\pi}'(i)}) = \min_{\pi \in \mathcal{S}_n} \sum_{i=1}^n D^2(B_i, R'_{\pi(i)})\right) \\
&= \sum_{i=1}^n |B_i - \Lambda R_{\hat{\pi}'(i)}|^2 = \sum_{i=1}^n \left| \begin{pmatrix} B_{i,x} \\ B_{i,y} \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 - \frac{1}{\sqrt{n}} \end{pmatrix} \begin{pmatrix} R_{\hat{\pi}'(i),x} \\ R_{\hat{\pi}'(i),y} \end{pmatrix} \right|^2 \\
&= \sum_{i=1}^n \left((B_{i,x} - R_{\hat{\pi}'(i),x})^2 + (B_{i,y} - R_{\hat{\pi}'(i),y} + \frac{1}{\sqrt{n}} R_{\hat{\pi}'(i),y})^2 \right) \\
&= \sum_{i=1}^n |B_i - R_{\hat{\pi}'(i)}|^2 + 2 \sum_{i=1}^n \frac{1}{\sqrt{n}} R_{\hat{\pi}'(i),y}(B_{i,y} - R_{\hat{\pi}'(i),y}) + \sum_{i=1}^n \frac{R^2_{\hat{\pi}'(i),y}}{n} \\
&= \sum_{i=1}^n |B_i - R_{\hat{\pi}'(i)}|^2 + \frac{1}{\sqrt{n}} \sum_{i=1}^n 2R_{\hat{\pi}'(i),y} B_{i,y} + (\frac{1}{n} - \frac{2}{\sqrt{n}}) \sum_{i=1}^n R^2_{\hat{\pi}'(i),y} \\
&= \sum_{i=1}^n |B_i - R_{\hat{\pi}'(i)}|^2 - \frac{1}{\sqrt{n}} \sum_{i=1}^n \left((R_{\hat{\pi}'(i),y} - B_{i,y})^2 - R^2_{\hat{\pi}'(i),y} - B^2_{i,y} \right) + (\frac{1}{n} - \frac{2}{\sqrt{n}}) \sum_{i=1}^n R^2_{\hat{\pi}'(i),y} \\
&= (1 - \frac{1}{\sqrt{n}}) \sum_{i=1}^n |B_i - R_{\hat{\pi}'(i)}|^2 + \frac{1}{\sqrt{n}} \sum_{i=1}^n B^2_{i,y} + (\frac{1}{n} - \frac{1}{\sqrt{n}}) \sum_{i=1}^n R^2_{\hat{\pi}'(i),y} \\
&\geqslant (1 - \frac{1}{\sqrt{n}}) \sum_{i=1}^n |B_i - R_{\hat{\pi}(i)}|^2 + \frac{1}{\sqrt{n}} \sum_{i=1}^n B^2_{i,y} + (\frac{1}{n} - \frac{1}{\sqrt{n}}) \sum_{i=1}^n R^2_{\hat{\pi}(i),y} \\
&\qquad \left(\text{here, we denote } \hat{\pi} \text{ such that } \sum_{i=1}^n D^2(B_i, R_{\hat{\pi}(i)}) = \min_{\pi \in \mathcal{S}_n} \sum_{i=1}^n D^2(B_i, R_{\pi(i)})\right) \\
&= (1 - \frac{1}{\sqrt{n}}) \mathcal{H}_{\text{opt}}^{\Omega,\Omega} + \frac{1}{\sqrt{n}} \sum_{i=1}^n B^2_{i,y} + (\frac{1}{n} - \frac{1}{\sqrt{n}}) \sum_{i=1}^n R^2_{\hat{\pi}(i),y}.
\end{aligned}
$$

Then, taking expectation of the two sides of the inequality, we get

$$
\begin{aligned}
\mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega,\Omega'}] &\geqslant (1 - \frac{1}{\sqrt{n}}) \mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega,\Omega}] + \mathbb{E}[\frac{1}{\sqrt{n}} \sum_{i=1}^n B^2_{i,y} + (\frac{1}{n} - \frac{1}{\sqrt{n}}) \sum_{i=1}^n R^2_{\hat{\pi}(i),y}] \\
&= (1 - \frac{1}{\sqrt{n}}) \mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega,\Omega}] + \mathbb{E}[\sqrt{n} B^2_{i,y}] + \mathbb{E}[(1 - \sqrt{n}) R^2_{\hat{\pi}(i),y}] \\
&= (1 - \frac{1}{\sqrt{n}}) \mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega,\Omega}] + \mathbb{E}[R^2_{\hat{\pi}(i),y}] \\
&= (1 - \frac{1}{\sqrt{n}}) \mathbb{E}[\mathcal{H}_{\text{opt}}^{\Omega,\Omega}] + \frac{1}{3}, \qquad \left(\text{here, we remark that } \mathbb{E}[R^2_{\hat{\pi}(i),y}] = \int_0^1 x^2 \mathrm{d}x = \frac{1}{3}\right).
\end{aligned}
$$

Thus, in particular,

$$\lim_{n \to +\infty} \mathbb{E}[\mathcal{H}_{\mathrm{opt}}^{\Omega,\Omega'}] \geqslant \lim_{n \to +\infty} \mathbb{E}[\mathcal{H}_{\mathrm{opt}}^{\Omega,\Omega}].$$

B Figures

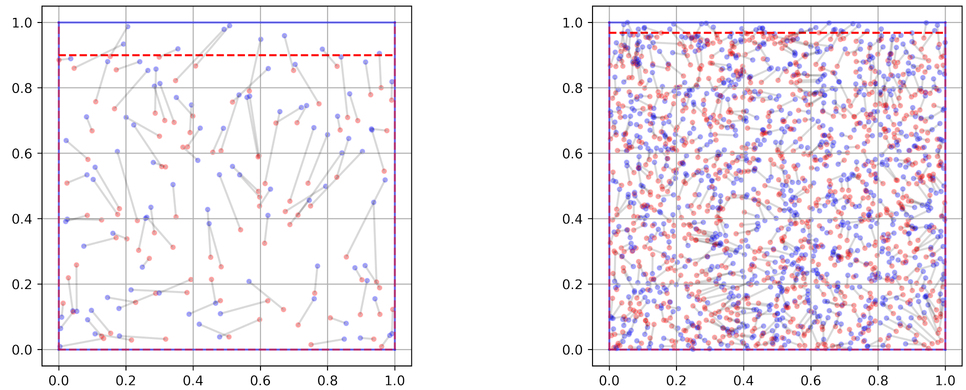

Figure B.1: Instances of case 1: $\Omega = [0,1]^2$ and $\Omega' = [0,1] \times [0, 1 - \frac{1}{\sqrt{n}}]$. Left: $n = 100$. Right: $n = 1000$.

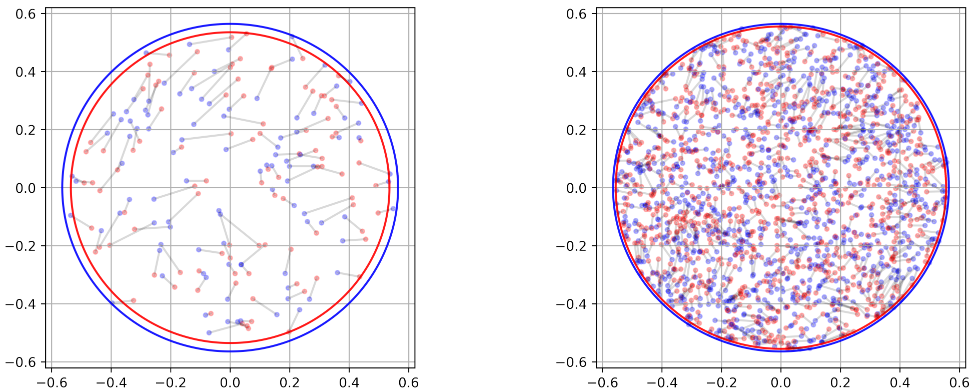

Figure B.2: Instances of case 2: $\Omega = B(0, \frac{1}{\sqrt{\pi}})$ and $\Omega' = B(0, \frac{1}{\sqrt{\pi}}\sqrt{1 - \frac{1}{\sqrt{n}}})$. Left: $n = 100$. Right: $n = 1000$.

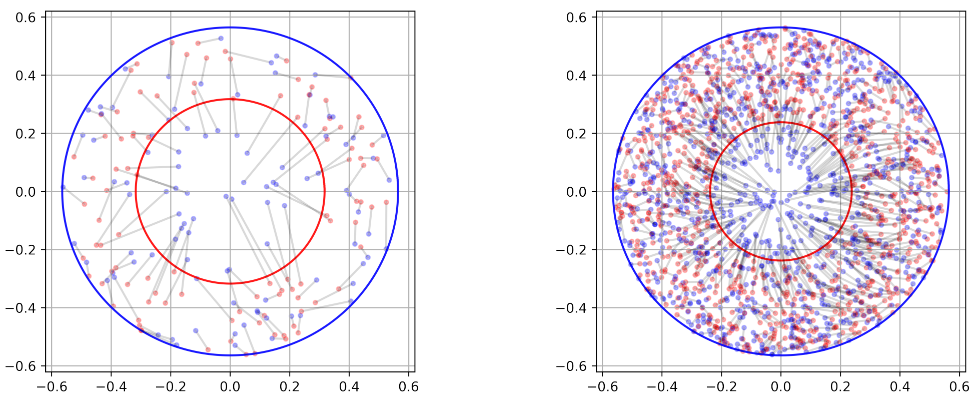

Figure B.3: Instances of case 3: $\Omega = B(0, \frac{1}{\sqrt{\pi}})$ and $\Omega' = B(0, \frac{1}{\sqrt{\pi}}) \setminus B(0, \frac{n^{-1/4}}{\sqrt{\pi}})$. Left: $n = 100$. Right: $n = 1000$.